Fourier analysis of wavelet algorithms for PDE's

Wavelets and differencial operators

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PhD Thesis: Wavelets for the simulation of Navier-Stokes in Turbulence
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Numerical Analysis Seminar - Warsaw - 11th october 2006

Outline

1. Wavelets to solve PDE's with local methods
   - Resolution of elliptic PDE's with wavelets
   - Shannon point of view
2. Two examples for Navier-Stokes equations
   - Implicit Laplacian
   - Leray projector
3. Numerical experiments
   - Observed convergence rates
   - Numerical simulations of Navier-Stokes

Outline

Elliptic operator equations [A. Cohen]

We want to solve

$$Af = g$$

where A is a linear differential operator and f the unknown function.
We use a wavelet expansion, an approximation A_j of A and a preconditionner D_j and write the sequence (f^n) :

$$f^n = f^{n-1} + D_j^{-1}(g - A_j f^{n-1})$$

Then this method converges if

$$\|f - f^n\| < \rho \|f - f^{n-1}\|$$

with $\rho < 1$

Operator theory basic elements [Lars Hörmander]

Differential operator of order m on a $\mathbb{R}^d$ functional space:

$$A = \sum_{|\alpha|=0}^{m} a_\alpha(x) D^\alpha$$

with $D^\alpha = (-i\partial_{x_1})^{\alpha_1} \dots (-i\partial_{x_d})^{\alpha_d}$
With the Fourier transform:

$$(Af)(x) = \int \frac{d\xi}{(2\pi)^d} e^{i\xi\cdot\mathbf{x}} \left(\sum_{|\alpha|=0}^{m} a_\alpha(x)\xi^\alpha \right) \widehat{f}(\xi)$$

Symbol: $\sigma(x,\xi) = \sum_{|\alpha|=0}^{m} a_\alpha(x)\xi^\alpha$
Principal symbol: $\sigma_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$

Operator theory basic elements

- elliptic operators:

$$\forall x \in \mathbb{R}^d, \forall \xi \in \mathbb{R}^d \setminus \{(0,\dots,0)\}, \sigma_m(x,\xi) \neq 0$$

- differential operators with constant coefficients: $\sigma(\xi)$
- pseudo-differential operators defined by the function $\sigma_m(x,\xi)$ under certain conditions

Linear operators expressed in Fourier

For **u** having several components, the symbol $\sigma(\xi)$ is a matrix $M(\xi)$. And we solve

$$M(\xi)\widehat{\mathbf{u}} = \widehat{\mathbf{v}} \qquad \text{or} \qquad \widehat{\mathbf{u}} = M(\xi)^{\dagger}\widehat{\mathbf{v}}$$

with $\mathbf{u} \in (H^t(\mathbb{R}^d))^m$, $\mathbf{v} \in (H^s(\mathbb{R}^d))^n$, $M(\xi) \in \mathcal{M}_{n\,m}(\mathbb{C})$ and $M(\xi)M(\xi)^{\dagger} = Id$.
Remark: $M(\xi)^{\dagger} = M(\xi)^{-1}$ if $m = n$.

If $\widehat{\mathbf{v}}$ and $\widehat{\mathbf{u}}$ are compactly supported, we build a matrix M_ω ($\omega \in \mathbb{R}^d$ depending on the compact support) such that :

$$M_\omega \approx M(\xi)$$

Linear operators expressed in Fourier

Then if we put the sequences $(\mathbf{u}_n)$ and $(\mathbf{v}_n)$:

$$\mathbf{u}_n = M_\omega^\dagger \mathbf{v}_{n-1} \quad \text{and} \quad \mathbf{v}_n = \mathbf{v}_{n-1} - M(\xi)\mathbf{u}_n$$

with $\mathbf{v}_0 = \mathbf{v}$.
It converges if

$$\exists \rho < 1 \text{ such that } \forall \xi \in \operatorname{supp}(\widehat{\mathbf{v}_n}),\ \|Id - M(\xi)M_\omega^\dagger\| < \rho.$$

The solution then writes:

$$\mathbf{u} = \sum_{n \leq 0} \mathbf{u}_n$$

Outline

Biorthogonal wavelets

Two scale filters m **and** n : functions $\varphi(\frac{\cdot}{2})$ and $\psi(\frac{\cdot}{2})$ remain to V_0, then there exist two sequences (a_k) and (b_k) verifying :

$$\varphi(\frac{x}{2}) = \sum_{k\in\mathbb{Z}} a_k \varphi(x-k) \quad , \qquad \psi(\frac{x}{2}) = \sum_{k\in\mathbb{Z}} b_k \varphi(x-k)$$

in Fourier,

$$\hat{\varphi}(2\xi) = m(\xi)\hat{\varphi}(\xi) \quad , \qquad \hat{\psi}(2\xi) = n(\xi)\hat{\varphi}(\xi)$$

with $m(\xi) = \frac{1}{2}\sum_{k\in\mathbb{Z}} a_k e^{-ik.\xi}$, $n(\xi) = \frac{1}{2}\sum_{k\in\mathbb{Z}} b_k e^{-ik.\xi}$
The scale function is inferred from the filter by :

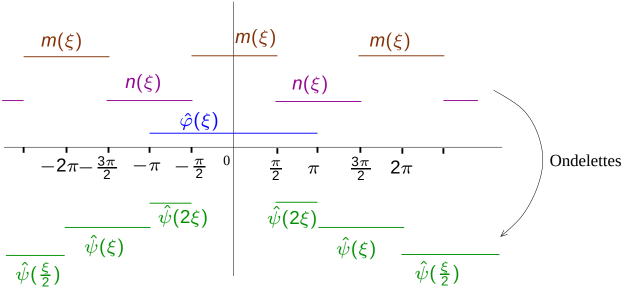
$$\hat{\varphi}(\xi) = \hat{\varphi}(0)\prod_{j=1}^{\infty} m(\frac{\xi}{2^j})$$

Shannon wavelets [S. Mallat]

Low-pass and high-pass perfect filters :

$$|m(\xi)| = \begin{cases} 1 & \text{if} \quad \xi \in [-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi], \quad k \in \mathbb{Z} \\ 0 & \text{else} \end{cases}$$

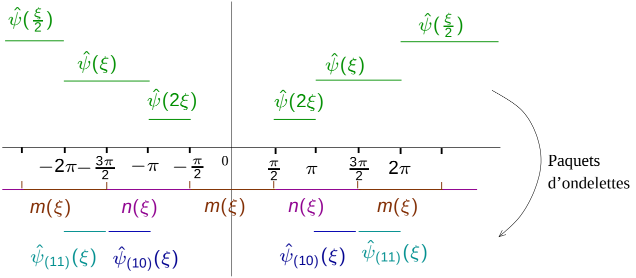
$$|n(\xi)| = \begin{cases} 1 & \text{if} \quad \xi \in [\frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi], \quad k \in \mathbb{Z} \\ 0 & \text{else} \end{cases}$$



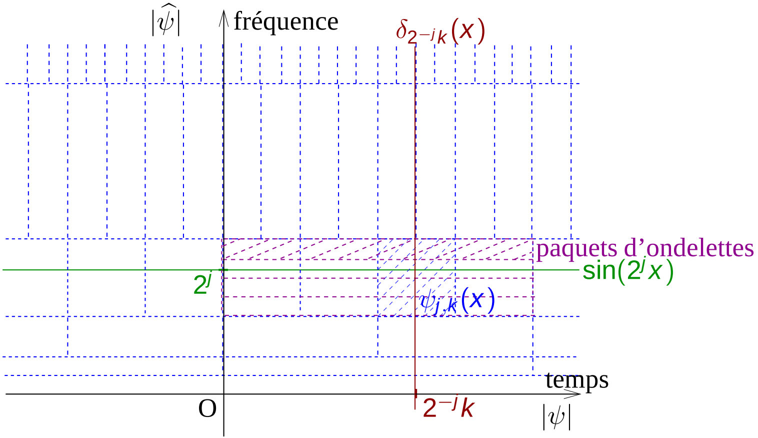
Shannon wavelet pakets

functions : $\hat{\varphi}(\xi) = \chi_{[-\pi,\pi]}(\xi)$ $\varphi(x) = \dfrac{\sin \pi x}{\pi x}$

$\hat{\psi}(\xi) = -e^{-i\xi/2}\chi_{[-2\pi,-\pi]\cup[\pi,2\pi]}(\xi)$ $\psi(x) = \dfrac{\sin 2\pi(x-1/2)}{2\pi(x-1/2)} - \dfrac{\sin \pi(x-1/2)}{\pi(x-1/2)}$



Time/frequency discretisation



Tensor Shannon decomposition

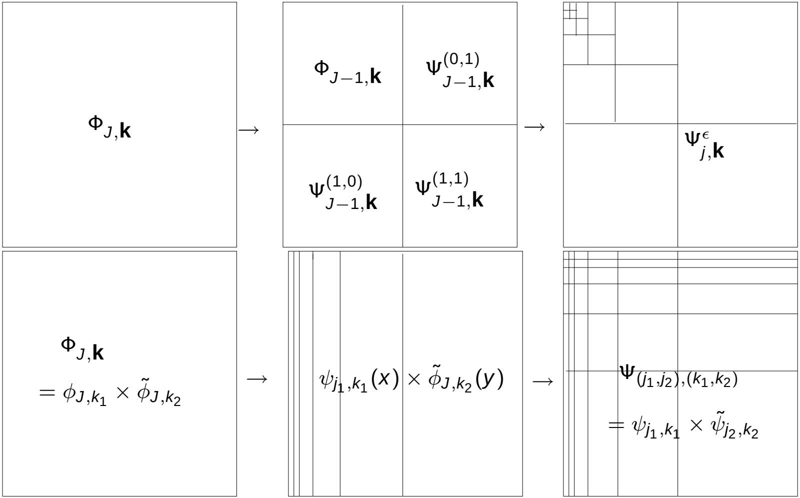
Let $\mathbf{u} : \mathbb{R}^d \to \mathbb{R}^m$
Shannon decomposition : $\mathbf{u} = \sum_{\mathbf{j}\in\mathbb{Z}^d} \mathbf{u_j}$
with

$$\mathrm{supp}\ \widehat{\mathbf{u}}_{\mathbf{j}} \subset \prod_{i=1}^{d} [-2^{j_i+1}\pi, -2^{j_i}\pi] \cup [2^{j_i}\pi, 2^{j_i+1}\pi]$$

For each $\mathbf{j} \in \mathbb{Z}^d$, and for $\ell = 1..m$,

$$u_{\mathbf{j},\ell} = \sum_{\mathbf{k}\in\mathbb{Z}^d} d_{\ell\,k}\psi_1(2^{j_1}x_1 - k_1)\ldots\psi_i(2^{j_i}x_i - k_i)\ldots\psi_n(2^{j_n}x_n - k_n)$$

Isotropic vs anisotropic



Convergence theorems with Shannon wavelets

Theorem (Anisotropic wavelets)

If the symbol matix $M(\xi)$ *is invertible* $\forall \xi \notin \{\xi_1 \dots \xi_d = 0\}$ *and* $\exists \rho < 1$ *such that* $\forall \mathbf{j} \in \mathbb{Z}^d \quad \exists M_{\omega_\mathbf{j}}$ *invertible such that* $\forall \xi \in \prod_{i=1}^d [2^{j_i}, 2^{j_i+1}], \quad \|Id - M(\xi)M_{\omega_\mathbf{j}}^{-1}\| \leq \rho$ *then the wavelet method converges.*

Theorem (Isotropic wavelets)

If the symbol matix $M(\xi)$ *is invertible* $\forall \xi \neq (0, \dots, 0)$ *and* $\exists \rho < 1$ *such that* $\forall j \in \mathbb{Z}$ *and* $\forall \varepsilon \in \{0,1\}^d \setminus \{(0, \dots, 0)\} \ \exists M_{\omega_j, \varepsilon}$ *invertible such that* $\forall \xi \in \prod_{i=1}^d [\varepsilon_i 2^j, (\varepsilon_i + 1)2^j]$, $\|Id - M(\xi)M_{\omega_j,\varepsilon}^{-1}\| \leq \rho$ *then the wavelet method converges.*

Outline

Iterative algorithm for $(Id - \alpha\Delta)\mathbf{u} = \mathbf{v}$

We want to solve $(Id - \alpha\Delta)\mathbf{u} = \mathbf{v}$ (in Navier-Stokes $\alpha = \nu\delta t$).
Shannon decomposition :

$$(Id - \alpha\Delta)\sum_{j\in\mathbb{Z}^d}\mathbf{u}_j = \sum_{j\in\mathbb{Z}^d}\mathbf{v}_j \text{ then } (Id - \alpha\Delta)\mathbf{u}_j = \mathbf{v}_j \text{ foreach} \mathbf{j} \in \mathbb{Z}^d$$

We approximate $M(\xi) = \frac{1}{1+\alpha|\xi|^2} Id$ by $M_\omega = \frac{1}{1+\alpha\omega^2} Id$, with $\xi \in \prod_{i=1}^d \pm[2^{j_i+1}\pi, 2^{j_i}\pi]$ and $\omega \in \mathbb{R}$ to be fixed later.

Iterative algorithm for $(Id - \alpha\Delta)\mathbf{u} = \mathbf{v}$

We have

$$\lambda(Id - M(\xi)^{-1}M_\omega) = 1 - \frac{1+\alpha|\xi|^2}{1+\alpha\omega^2}$$

that remains minimal for $\omega^2 = \frac{a^2+b^2}{2}$ if $|\xi|^2 \in [a^2, b^2]$.

Then $|\lambda(\xi)| \leq \frac{\alpha(b^2-a^2)}{2+\alpha(b^2+a^2)}$

for $a = 1$, $b = 2$ (Shannon) and $\alpha \to \infty$, the rate is $\frac{3}{5}$ (0.6).

for $a = 1$, $b = \sqrt{2}$ (wavelet pakets), it is $\frac{1}{3}$ (0.33).

Outline

Principle of Helmholtz decomposition

Vector field $\mathbf{u} \in (L^2(\mathbb{R}^n))^n$, decomposition with

$$\mathbf{u} = \mathbf{u}_{\mathrm{div}} + \mathbf{u}_{\mathrm{curl}} \quad \text{where} \quad \mathbf{u}_{\mathrm{div}} = \mathbf{curl}\,\psi \quad \mathbf{u}_{\mathrm{curl}} = \nabla p$$

the functions $\mathbf{curl}\,\psi$ and ∇p are orthogonal in $(L^2(\mathbb{R}^n))^n$ and are unique.

$$(L^2(\mathbb{R}^n))^n = \mathbf{H}_{\mathrm{div}\,0}(\mathbb{R}^n) \oplus^{\perp} \mathbf{H}_{\mathrm{curl},0}(\mathbb{R}^n)$$

In N-S, importance of this decomposition to project the term $\mathbf{u}.\nabla\mathbf{u}$ onto $\mathbf{H}_{\mathrm{div}\,0}(\mathbb{R}^n)$.

Iterative algorithm for $\mathbb{P}$

We approximate the Leray projector $\mathbb{P}$: $\mathbb{P}\widehat{\mathbf{u}} = M(\xi)\widehat{\mathbf{u}}$, with

$$M(\xi) = Id - \frac{1}{|\xi|^2}\begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \times [\xi_1 \dots \xi_n]$$

using wavelets s.t. $\psi_1'(x) = \psi_0(x)$ i.e. $i\xi\widehat{\psi_1}(\xi) = \widehat{\psi_0}(\xi)$, by $M_\omega =$

$$\left(Id - \frac{1}{|\omega|^2}\begin{bmatrix} \frac{\omega_1^2}{\xi_1} \\ \vdots \\ \frac{\omega_n^2}{\xi_n} \end{bmatrix} \times [\xi_1 \dots \xi_n]\right)\left(Id - \frac{1}{|\omega|^2}\begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \times \left[\frac{\omega_1^2}{\xi_1} \dots \frac{\omega_n^2}{\xi_n}\right]\right)$$

Iterative algorithm for $\mathbb{P}$

We also have to extract the gradient part.
We approximate $\nabla\Delta^{-1}(\mathrm{div}\cdot)$ by

$$L_\omega = \frac{1}{|\omega|^2}\begin{bmatrix}\xi_1\\ \vdots \\ \xi_n\end{bmatrix} \times \begin{bmatrix}\frac{\omega_1^2}{\xi_1} \ldots \frac{\omega_n^2}{\xi_n}\end{bmatrix}$$

then

$$\mathbf{v}_n = \mathbf{v}_{n-1} - M_\omega \mathbf{v}_{n-1} - L_\omega \mathbf{v}_{n-1}$$

Iterative algorithm for $\mathbb{P}$

We find

$$\lambda = 1 - \left(\sum_{k=1}^{n} \xi_k^2\right)\left(\sum_{k=1}^{n} \frac{\omega_k^4}{|\omega|^4 \xi_k^2}\right)$$

i.e.

$$\lambda = 1 - \left(\sum_{k=1}^{n} \frac{\omega_k^2}{|\omega|^2} x_k^2\right)\left(\sum_{k=1}^{n} \frac{\omega_k^2}{|\omega|^2} x_k^{-2}\right)$$

with $x_k = \frac{\xi_k}{\omega_k}$.
Thanks to Kantorovitch inequality,

$$|\lambda|_{\max} \leq \frac{1}{4}\left(\frac{min\ |x_k|}{max\ |x_k|} + \frac{max\ |x_k|}{min\ |x_k|}\right)^2 - 1$$

Iterative algorithm for $\mathbb{P}$

If $|x_k| \in [a_k, b_k]$ for every k, we put $a = \min_k\ a_k$ and $b = \max_k\ b_k$. Then

$$|\lambda| \leq \frac{1}{4}\left(\frac{a}{b} + \frac{b}{a}\right)^2 - 1$$

for $a = 1$ and $b = 2$ (Shannon), $|\lambda| \leq \frac{9}{16}$ (~ 0.56). Then convergence.
for $a = 1$ and $b = \frac{3}{2}$ (wavelet pakets), $|\lambda| \leq \frac{25}{144}$ (~ 0.17).

Outline

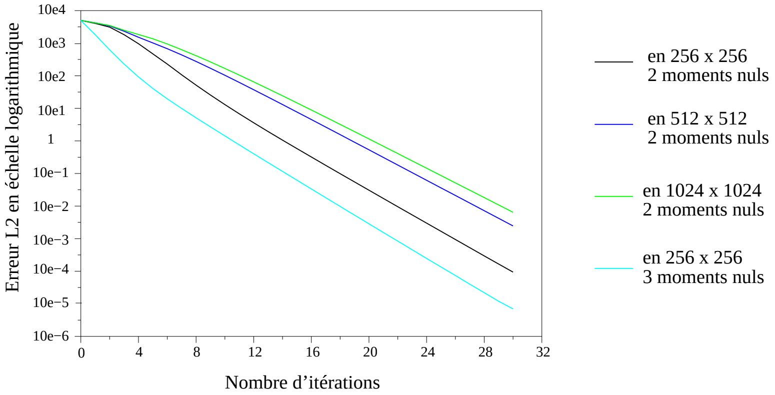
Convergence for the implicit Laplacian

For $Id - \alpha\Delta$, the convergence is very fast if α is small compared to the smallest computed scale.

For the Laplacian Δ ($\alpha \to +\infty$), the observed convergence is around 0.75.

Numerical Tests of the convergence for Leray

Numerically, the convergence has been tested successfully on variate 2D and 3D fields. Average convergence rate: 0.5



Outline

$\mathbb{P}$ Leray projector with wavelets.
Entails the pressure:

$$[\mathbf{u}.\nabla \mathbf{u}]_{\mathrm{curl}} = -\nabla p = \sum_{\mathbf{j},\mathbf{k}} d_{\mathrm{rot}\,\mathbf{j},\mathbf{k}} \frac{1}{4} \nabla \left(\psi_1(2^{j_1}x_1 - k_1)\psi_1(2^{j_2}x_2 - k_2)\right)$$

We treat the Laplacian Δ implicitly with a wavelet preconditionner.

Semi-implicit schema in time:

$$\mathbf{u}^{n+1} = \mathbf{u}^n - \delta t \mathbb{P}\,[\mathbf{u}.\nabla \mathbf{u}]^n + \delta t \nu \Delta \mathbf{u}^{n+1}$$

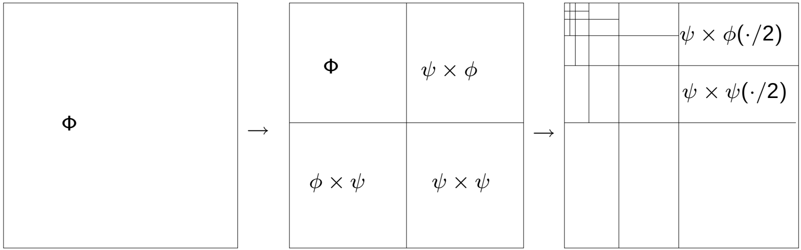
on the wavelet coefficients:

$$(Id - \nu \delta t \Delta) d_{i\,\mathbf{j},\mathbf{k}}^{\mathrm{div},n+1} = d_{i\,\mathbf{j},\mathbf{k}}^{\mathrm{div},n} - \delta t d_{i\,\mathbf{j},\mathbf{k}}^{\mathrm{div}}(\mathbb{P}\,[(\mathbf{u}\cdot\nabla)\mathbf{u}])$$

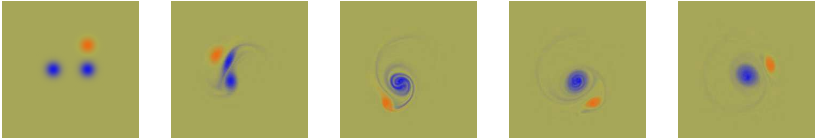
Numerical test on the "merging of 3 vortices"

- simplest spline of degree 1 and 2 **wavelet code**
- *semi-implicit* schema of order 2 for the time evolution
- 256^2 grid, $\delta t = 0.02$ and $\nu = 5.10^{-5}$
- 14 iterations for Helmholtz, 4 for the implicit Laplacian,
- Code using **uniquely** wavelet transforms
- Pseudo-adaptive code

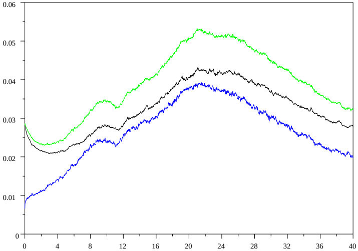
Partialy anisotropic wavelet transform



Numerical test on the "merging of 3 vortices"



t=0 t=10 t=20 t=30 t=40



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Conclusion - Perspectives

Assets

- Local computations in $O(n)$
- Time/frequency discretisation
- Local regularity estimate

Perspectives

- Theoretical proof for other wavelets than Shannon
- Study for other operators
- Maxwell equations
- Real adaptive code

Perspective: adaptive method

- **Numerical resolution** of incompressible Navier-Stokes equations in dimension 2 and 3 by an adaptive method:

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla \mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \mathbf{f} & t \geq 0,\ x \in]0,1[^n\ n = 2 \text{ or } 3 \\ \text{div}\ \mathbf{u} = \nabla \cdot \mathbf{u} = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 & \\ \mathbf{u}(x,0) = \mathbf{u}^0(x) & \end{cases}$$

With an adaptive wavelet discretization:

$$\mathbf{u}(x,t^n) \approx \mathbf{u}_N(x,t^n) = \sum_{\alpha \in A_n} c_\alpha^n\ \Psi_\alpha(x)$$

with $Card(A_n) = N$ and $\Psi_\alpha \in \mathrm{H}_{\mathrm{div},0} = \{\mathbf{u} \in L^2 / \mathrm{div}(\mathbf{u}) = 0\}$.

Perspective: operators with non constant coefficients

$$M(x,\xi) = B(x)C(\xi)$$

Approximated by: $A_{a,\omega} = B_a C_\omega$

Use the decomposition:

$$Id - C^{-1}B^{-1}B_a C_\omega = (Id - C^{-1}C_\omega) + C^{-1}(Id - B^{-1}B_a)C_\omega$$

But, it is difficult since $B^{-1}B_a$ creates undesired frequencies.