Numerical approximation of multivariate differential operators using multiscale preconditioners and wavelet differentiation

Erwan Deriaz * , Jacques Liandrat †

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Abstract

Wavelet decompositions and wavelet differentiation are coupled to derive new algorithm of multigrid type for the computation of inverse operators. These methods can be easily analyzed in term of convergence rate when the Shannon wavelets are used. Different applications are developed. They concern elliptic equations, Helmholtz decomposition and Craya decomposition.

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1 Introduction

Since their introduction at the end of the 80's, *Wavelets* have provided new numerical methods for partial differential equations (PDE's). Thanks to the Fast Wavelet Transform (FWT) and to the sparsity of the wavelet mass matrices, efficient algorithms including optimal preconditioners for elliptic operators have been derived [19].

Wavelet approach also resulted in non-linear approximations [10], and in denoising methods [17]. Wavelets offer the possibility to conjugate the accuracy in space with the accuracy in frequency. This property leads to efficient algorithms for solving PDE's in an adaptive context. Cohen, Dahmen and De Vore enhanced the interest for these methods applied to the solution of PDE's. In [8, 9], they proved the optimal complexity of wavelet algorithms for the solution of elliptic problems.

This article is devoted to the definition and the analysis of a fast iterative method to solve various PDE's by approximating their inverse operators. It is based on specific wavelet bases and diagonal operators and can be easily implemented using *fast wavelet transforms* on adapted spaces of approximation. We address particularly the approximation of elliptic operators in multidimension and the approximation of Helmholtz and Craya projectors.

*Institute of Mathematics, Polish Academy of Sciences. ul. Sniadeckich 8, 00-956 Warszawa, Poland, to whom correspondence should be addressed (E.Deriaz@impan.gov.pl)

†Centrale Marseille/LATP, Technopôle de Château-Gombert, 38 rue F. Joliot Curie, 13451 Marseille Cedex 20, France (jliandrat@ec-marseille.fr)

Pioneering works on wavelet methods for the numerical approximation of partial differential equation solutions [19, 21, 25, 6] are, generally speaking, of Galerkin or Petrov-Galerkin type, exploiting as much as possible the efficiency of fast wavelet algorithms on sparse adapted space of approximation.

Even in the more recent works [8, 9], solving $Lu = f$ with boundary conditions, where L is a differential operator. generally involves a wavelet basis $\{\psi_{jk}\}_{j,k\in\mathbb{Z}}$ –where f_{jk} stands for $2^{j/2}f(2^jx-k)$– and associated families. These associated families which are $\{L^{-1}\psi_{jk}\}_{j,k\in\mathbb{Z}}$, $\{L\psi_{jk}\}_{j,k\in\mathbb{Z}}$ or $\{\Lambda_j\psi_{jk}\}_{j,k\in\mathbb{Z}}$ where Λ_j stands for a preconditioning family of reals (for a second order differential operator $\Lambda_j = 4^{-j}$), belong to the class of vaguelette families that ensure, theoretically but not always in practice the existence of efficient associated transform algorithms.

An other work in the background of this paper is the article of P.G. Lemarié [22] on differentiation of wavelet bases that shows that the above families of vaguelettes are indeed biorthogonal families of wavelets when L is an homogeneous differential operator.

Our work exploits the framework of biorthogonal wavelets to construct efficient preconditioners that can be plugged in an iterative solution of a PDE We will show that our preconditioners are more efficient than the previously defined one and that the biorthogonal framework makes that the global solution algorithm outperform the others, especially when adaptivity in multidimension is required.

The organization of the article is as follows: in section 2, we recall some wavelet theory basic elements, and we introduce a result on the differentiation of biorthogonal wavelets due to P. G. Lemarié-Rieusset [22].

In section 3, following the works [25, 8] on the wavelet approximation of differential operators, we review the mechanism of the Richardson iteration with wavelet preconditioning. We also state the correlated theorem of convergence.

In section 4, we recall some basic facts regarding operator theory and relate them to wavelet decomposition. Then, we define the family of all the operators that constructible using wavelet differentiation. This leads to the construction of new wavelet approximations for differential operators and projection operators.

In section 2.2, the case of Shannon wavelet is precised and we give accurate estimate for the convergence rate of the Richardson iteration in this case. We also indicate how to prove the convergence of the method in the general case.

Finally, in section 6 we apply our results in two situations. First, we consider the inverse Laplace operator Δ^{-1} and derive new wavelet solvers for the Laplace equation. Second we consider Helmholtz and Craya decompositions.

2 Biorthogonal wavelet bases and wavelet differentiation

The material presented in this section comes from the academic book [20].

2.1 General construction of biorthogonal wavelets

Biorthogonal wavelets are basically defined through two couples of 2π-periodic functions so-called **scale filters** $(m(\xi), n(\xi))$ and $(m^*(\xi), n^*(\xi))$ that allow to define all together the wavelets and the associated scaling functions.

If $m(\xi) = \frac{1}{2}\sum_{k\in\mathbb{Z}} a_k e^{-ik.\xi}$ and $m^*(\xi) = \frac{1}{2}\sum_{k\in\mathbb{Z}} a_k^* e^{-ik.\xi}$ then the scaling functions φ

and φ^* are defined by their Fourier transform[1]:

$$\hat{\varphi}(\xi) = \hat{\varphi}(0) \prod_{j=1}^{\infty} m(\frac{\xi}{2^j}), \quad \hat{\varphi^*}(\xi) = \hat{\varphi^*}(0) \prod_{j=1}^{\infty} m^*(\frac{\xi}{2^j})$$

and if $n(\xi) = e^{-i\xi}\overline{m^*(\xi+\pi)}$ and $n^*(\xi) = e^{-i\xi}\overline{m(\xi+\pi)}$ the wavelets are defined by:

$$\hat{\psi}(\xi) = \hat{\varphi}(0) \prod_{j=1}^{\infty} n(\frac{\xi}{2^j}), \quad \hat{\psi^*}(\xi) = \hat{\varphi^*}(0) \prod_{j=1}^{\infty} n^*(\frac{\xi}{2^j})$$

This corresponds to the scale relations:

$$\varphi(\frac{x}{2}) = \sum_{k \in \mathbb{Z}} a_k \varphi(x-k) \quad , \qquad \psi(\frac{x}{2}) = \sum_{k \in \mathbb{Z}} (-1)^{k+1} a^*_{1-k} \varphi(x-k)$$

As soon as $(m(\xi), n(\xi))$ and $(m^*(\xi), n^*(\xi))$ satisfy specific conditions (see [20]), then the wavelet families $\{\psi_{jk}\}_{j,k\in\mathbb{Z}}$ and $\{\psi^*_{jk}\}_{j,k\in\mathbb{Z}}$ form a dual couple of biorthogonal Riesz basis of $L^2(\mathbb{R})$:

$$f = \sum \langle f, \psi^*_{jk} \rangle \psi_{jk} = \sum \langle f, \psi_{jk} \rangle \psi^*_{jk}$$

The spaces $V_j = \text{span}\{\varphi_{jk}\}_{k\in\mathbb{Z}}$ and $V^*_j = \text{span}\{\varphi^*_{jk}\}_{k\in\mathbb{Z}}$ form a biorthogonal multiresolution of $L^2(\mathbb{R})$.

Moreover, if $\dot{H}^t(\mathbb{R})$ stands for the homogeneous Sobolev space $\dot{H}^t(\mathbb{R}) = \{|\xi|^t \widehat{f} \in L^2\}$ then if

$$t_1 = \sup\{t \in \mathbb{R}, \psi^* \in \dot{H}^t(\mathbb{R})\} \ \text{ and } \ t_2 = \sup\{t \in \mathbb{R}, \psi \in \dot{H}^t(\mathbb{R})\}$$

the family $\{2^{tj}\psi_{jk}\}_{j,k\in\mathbb{Z}}$ is a Riesz basis of $\dot{H}^t(\mathbb{R})$ for $-t_1 \le t \le t_2$.
We denote by ℓ^2_t the space of sequences $(u_{jk})_{(j,k)\in\mathbb{Z}^2}$ with the norm $\|(u_{jk})_{(j,k)\in\mathbb{Z}^2}\|^2_{\ell^2_t} = \sum_{(j,k)\in\mathbb{Z}^2} 2^{2tj}|u_{jk}|^2$. The scale decomposition (3.10) operated by wavelets is characterized by the coefficients $B_t, b_t > 0$ which are such that $\forall (u_{jk})_{j,k\in\mathbb{Z}} \in \ell^2_t$,

$$b_t \sum_{j\in\mathbb{Z}} \| \sum_{k\in\mathbb{Z}} u_{jk}\psi_{jk}\|^2_{\dot{H}^t} \le \| \sum_{j,k\in\mathbb{Z}} u_{jk}\psi_{jk}\|^2_{\dot{H}^t} \le B_t \sum_{j\in\mathbb{Z}} \| \sum_{k\in\mathbb{Z}} u_{jk}\psi_{jk}\|^2_{\dot{H}^t}$$

If $b_t = B_t = 1$, the wavelet basis is said semi-orthogonal in $\dot{H}^t$.

2.2 The Shannon wavelet

The definition of the Shannon wavelets can be found in Mallat's academic book [23]. Here we state the essential properties of these wavelets.

Shannon wavelets are particular in the meaning that they have perfect low-pass and high-pass filters :

$$m(\xi) = \begin{cases} 1 & \text{if} \quad \xi \in [-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi], \quad k \in \mathbb{Z} \\ 0 & \text{if} \quad \xi \in [\frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi], \quad k \in \mathbb{Z} \end{cases} \tag{2.1}$$

$$n(\xi) = \begin{cases} -e^{-i\xi} & \text{if} \quad \xi \in [\frac{\pi}{2} + 2k\pi, \frac{3\pi}{2} + 2k\pi], \quad k \in \mathbb{Z} \\ 0 & \text{if} \quad \xi \in [-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi], \quad k \in \mathbb{Z} \end{cases} \tag{2.2}$$

Then the corresponding scaling function writes:

[1] The Fourier transform of a function $f \in L^1(\mathbb{R})$ is noted $\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x)\, e^{-ix\xi} dx$

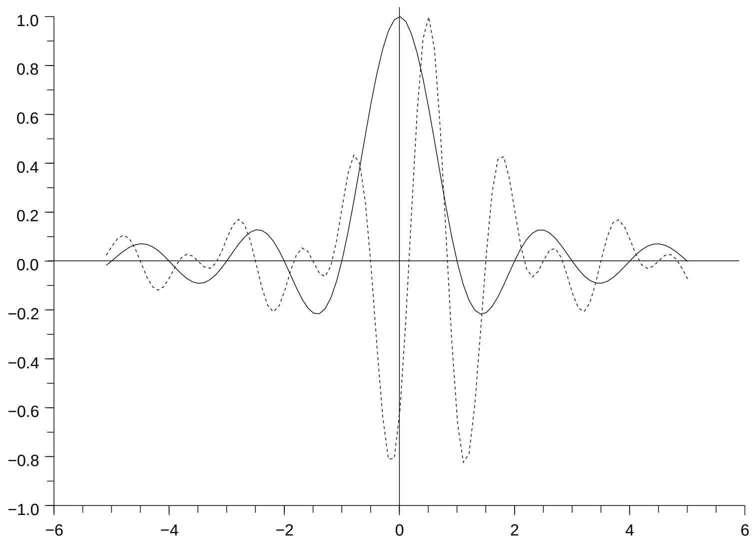

Figure 1: Shannon scaling function (plain) and wavelet (dotted).

$$\hat{\varphi}(\xi) = \chi_{[-\pi,\pi]}(\xi) \qquad , \qquad \varphi(x) = \frac{\sin \pi x}{\pi x}$$

and the wavelet:

$$\hat{\psi}(\xi) = e^{-i\xi/2}\ \chi_{[-2\pi,-\pi]\cup[\pi,2\pi]}(\xi) \qquad , \qquad \psi(x) = \frac{\sin 2\pi(x-1/2)}{\pi(x-1/2)} - \frac{\sin \pi(x-1/2)}{\pi(x-1/2)}$$

where χ stands for the characteristic function i.e. $\chi_E(x) = 1$ if $x \in E, 0$ if $x \notin E$. Hence for Shannon wavelets, $B_t = b_t = 1\ \forall t \in \mathbb{R}$.

In the multidimensional case, the **tensorial Shannon decomposition** writes as follows:
Let $\mathbf{u} : \mathbb{R}^d \to \mathbb{R}^n$. The Shannon decomposition of $\mathbf{u}$ is given by:

$$\mathbf{u} = \sum_{\mathbf{j}\in\mathbb{Z}^d} \mathbf{u}_\mathbf{j} \tag{2.3}$$

with

$$\text{supp } \widehat{\mathbf{u}_\mathbf{j}} \subset \prod_{i=1}^{d} [-2^{j_i+1}\pi, -2^{j_i}\pi] \cup [2^{j_i}\pi, 2^{j_i+1}\pi]$$

For each scale parameter $\mathbf{j} \in \mathbb{Z}^d$, and for each component $\ell = 1 \dots n$, we have:

$$u_{\ell,\mathbf{j}}(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} 2^{|\mathbf{j}|/2} u_{\ell\mathbf{j}\mathbf{k}} \psi_1^\ell(2^{j_1}x_1 - k_1) \dots \psi_i^\ell(2^{j_i}x_i - k_i) \dots \psi_d^\ell(2^{j_d}x_d - k_d)$$

where $|\mathbf{j}| = \sum_i j_i$ and ψ_i^ℓ are wavelets of Shannon type, i.e. $\text{supp}\,\widehat{\psi_i^\ell} = [-2\pi, -\pi] \cup [\pi, 2\pi]$.

Remark 2.1 *As the Shannon wavelets are C^∞ and have an infinite number of zero moments, they can be differentiated or integrated in order to obtain biorthogonal wavelets satisfying the relations (2.10) of proposition 2.2 and which form an MRA of $L^2(\mathbb{R})$. We can iterate the differentiation or the integration of these wavelets as many times as we wish in order to obtain derivatives of arbitrary order: $\dots$, ψ_{-2}, ψ_{-1}, $\dots$, ψ_2, $\dots$, with ψ_0 the original Shannon wavelet.*

2.3 Shannon wavelet packets

With the filters $m(\xi)$ and $n(\xi)$ defined in section 2.2 (2.1) and (2.2), we can define the Shannon wavelet packets. The wavelet packets are defined by applying the filters m and n to the wavelets themselves. Hence we obtain two new wavelets $\psi_{(11)}$ and $\psi_{(10)}$ which are twice better localized in the Fourier space (i.e. the compact supports of their Fourier transforms are twice smaller):

$$\widehat{\psi_{(10)}}(2\xi) = m(\xi)\hat{\psi}(\xi) \tag{2.4}$$

$$\widehat{\psi_{(11)}}(2\xi) = n(\xi)\hat{\psi}(\xi) \tag{2.5}$$

The two of them are necessary to expend $L^2(\mathbb{R})$, i.e.

$$L^2(\mathbb{R}) = \overline{span\{\psi_{(10))}(2^j x - 2k), \psi_{(11)}(2^j x - 2k)\}_{j,k\in\mathbb{Z}}}$$

More precisely, the wavelet space at level j W_j admits $\{\psi_{(10))}(2^j x - 2k), \psi_{(11)}(2^j x - 2k)\}_{k\in\mathbb{Z}}$ as a Riesz basis.

The operations (2.4) and (2.5) can be iterated as many times as needed on the Shannon wavelet, so its Fourier support is sufficiently shrunk. In practice this operation can also be applied to usual wavelets but doesn't come out with good results. Getting a better frequency localization for usual wavelet packets is still a challenging problem.

The Shannon wavelet packet decomposition gives us the opportunity to approximate operators more closely than the classical Shannon decomposition (2.3). One can report to part 5.3 for more details.

2.4 Wavelet in multidimension

The simplest way to construct wavelets in multidimension is to perform a tensorial product of univariate multiresolution. This, however leads to two different types of bases: the tensorial (or *anisotropic*) wavelet bases are obtained by taking the tensorial product of univariate wavelet, while the multidimensional multiresolution wavelets are obtained from a multiresolution of $L^2(\mathbb{R}^d)$.
In other terms this corresponds to:

$$L^2(\mathbb{R}^d) = \text{clos}(\oplus_{\mathbf{j}\in\mathbb{Z}^d} W_{\mathbf{j}}) = \text{clos}(\oplus_{\mathbf{j}\in\mathbb{Z}^d} \otimes_{i=1}^d W_{j_i}) = \text{clos}(\otimes_{i=1}^d \oplus_{j_i\in\mathbb{Z}} W_{j_i})$$

for the tensorial wavelet basis. The corresponding wavelet decomposition and non-linear approximation are called hyperbolic because their graphs of coefficients in Fourier space are folded in hyperboles. And it corresponds to:

$$L^2(\mathbb{R}^d) = \text{clos}(\oplus_{j\in\mathbb{Z}} \mathbf{W}_j) \quad \text{with} \quad \mathbf{W}_j = \mathbf{V}_{j+1} \setminus \mathbf{V}_j = \oplus_{\varepsilon\in\{0,1\}^d\setminus 0} \otimes_{i=1}^d \mathcal{W}_j^{\varepsilon_i}$$

with $\mathcal{W}_j^0 = V_j$ and $\mathcal{W}_j^1 = W_j$, for the multiresolution wavelets.
We note $\{\psi_{\mathbf{jk}}\}_{\mathbf{j}\in\mathcal{J},\mathbf{k}\in\mathbb{Z}^d}$ the bases and $\mathcal{J} = \mathbb{Z}^d$ for tensorial wavelets while $\mathcal{J} = \mathbb{Z} \times \big(\{0,1\}^d \setminus 0\big)$ in the other case.

2.5 Vector valued wavelets

When $\mathbf{u}$ is a vector valued function: $\mathbf{u}(x) \in \mathbb{R}^d$, each component of $\mathbf{u}$ is decomposed on a vector wavelet basis. If we denote by $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ the canonical basis of $\mathbb{R}^n$, then $\{\psi_{\mathbf{jk}}\mathbf{e}_\ell\}_{\mathbf{j}\in\mathcal{J},\mathbf{k}\in\mathbb{Z}^d,\ell\in[1,n]}$ is a wavelet vector basis of $\left(L^2(\mathbb{R}^d)\right)^n$. With $\Psi_{\mathbf{jk}\ell} = \psi_{\mathbf{jk}}e_\ell$, writting

$$\mathbf{u} = \sum_{\mathbf{j}\in\mathcal{J},\mathbf{k}\in\mathbb{Z}^d,\ell\in[1,n]} \langle \mathbf{u}, \Psi^*_{\mathbf{jk}\ell}\rangle \Psi_{\mathbf{jk}\ell} = \sum_{\mathbf{j}\in\mathcal{J}} \mathbf{u_j}$$

$\mathbf{u_j}$ is the projection on the wavelet space $\mathbf{W_j} = span(\{\Psi_{\mathbf{jk}\ell}\}_{\mathbf{k}\in\mathbb{Z}^d,\ell\in[1,n]})$.

2.6 Wavelet derivatives

P. G. Lemarié-Rieusset [22] showed that differentiating or integrating a biorthogonal wavelet basis provides new wavelet bases. Starting from a Riesz basis of H^t [2] $H^t(\mathbb{R})$, this allows to construct two different multiresolution analyzes of $H^t(\mathbb{R})$ and $H^{t+1}(\mathbb{R})$ for some $t \in \mathbb{R}$, related by differentiation and integration. Indeed:

Proposition 2.1 *Let φ_0 a scale function and $m_0(\xi)$ its filter. Then*

(i) if $\varphi_0 \in \dot{H}^1$ then there exists a scale function φ_{-1} such that

$$\varphi'(x) = \varphi_{-1}(x) - \varphi_{-1}(x-1) \quad \textit{(differentiation formula)} \tag{2.6}$$

and the filter $m_{-1}(\xi)$ associated to φ_{-1} satisfies

$$m_{-1}(\xi) = \frac{2}{1+e^{-i\xi}} m_0(\xi). \tag{2.7}$$

(ii) There exists a scale function φ_1 such that

$$\varphi_0(x) - \varphi_0(x-1) = \varphi_1'(x) \quad \textit{(integration formula)} \tag{2.8}$$

and the filter $m_1(\xi)$ associated to φ_1 satisfies

$$m_1(\xi) = \frac{1+e^{-i\xi}}{2} m_0(\xi). \tag{2.9}$$

Proposition 2.2 (Differentiation of wavelets) *[22] Let $(V_{1\,j})_{j\in\mathbb{Z}}$ be a one-dimensional MRA, made up of a differentiable scaling function φ_1, ($V_{1\,0} = span\{\varphi_1(x-k), k \in \mathbb{Z}\}$), and a wavelet ψ_1. There exists a second MRA $(V_{0\,j})_{j\in\mathbb{Z}}$ with a scaling function φ_0 ($V_{0\,0} = span\{\varphi_0(x-k), k \in \mathbb{Z}\}$) and a wavelet ψ_0 satisfying:*

$$\varphi_1'(x) = \varphi_0(x) - \varphi_0(x-1) \quad \textit{and} \quad \varphi_1^*(x+1) - \varphi_1^*(x) = \varphi_0^{*'}(x) \tag{2.10}$$

The filters (m_0, m_0^*) and (m_1, m_1^*) (see part 2.2 for the definition of a filter) attached respectively to the MRAs $(V_{0\,j})_{j\in\mathbb{Z}}$ and $(V_{1\,j})_{j\in\mathbb{Z}}$ verify:

$$m_0(\xi) = \frac{2}{1+e^{-i\xi}}\, m_1(\xi) \quad \textit{and} \quad m_0^*(\xi) = \frac{1+e^{i\xi}}{2}\, m_1^*(\xi) \tag{2.11}$$

[2] The space H^t denotes the Sobolev space $\left(H^t(\mathbb{R}^d)\right)^n$ with $H^t(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{R},\ (1+|\xi|^2)^{t/2}|\hat{f}(\xi)| \in L^2(\mathbb{R}^d)\}$.

And the wavelets ψ_1 and ψ_0 associated respectively to spaces (W_j^1) and (W_j^0) satisfies:

$$\psi_1'(x) = 4\ \psi_0(x) \quad \textit{and} \quad \int \psi_1^*(x) = -\frac{1}{4}\psi_0^*(x) \tag{2.12}$$

Expressed with its Fourier transform this latter relation writes:

$$i\xi\widehat{\psi_1}(\xi) = 4\widehat{\psi_0}(\xi) \quad \textit{and} \quad \frac{1}{i\xi}\widehat{\psi_1^*}(\xi) = -\frac{1}{4}\widehat{\psi_0^*}(\xi) \tag{2.13}$$

If the wavelet ψ_1 is C^n and has p zero moments (i.e. $\widehat{\psi}_1$ is $p-1$ times differentiable and $\widehat{\psi}_1^{(k)}(0) = 0$ for $0 \leq k \leq p-1$) the wavelet derivative ψ_0 has regularity C^{n-1} and $p+1$ zero moments.

Remark 2.2 *Originally, in [22], this result was applied only to compactly supported wavelets. Nevertheless it can be extended to other wavelets, provided that $|\xi|\hat{\psi}_1(\xi) \in L^2$ and $|\xi|^{-1}\widehat{\psi_1^*}(\xi) \in L^2$. Alternatively, we consider MRAs of Hilbert spaces $H^t(\mathbb{R})$. Also one can verify that filters issued from relations (2.11) applied to Shannon filters (part 2.2) provide a MRA of $L^2(\mathbb{R})$.*

On account of the above remark, we can introduce a new operation thanks to the wavelet decomposition of a function v. If we use the tensorial wavelet decomposition, we can differentiate or integrate in every directions. For instance, if we write the wavelet decomposition of v with wavelets ψ_0 for each tensorial components except for i for which we take ψ_1 where $\psi_1' = 4\psi_0$ (relation (2.10) of proposition 2.2):

$$v(x) = \sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^d} d_{\mathbf{j}\,\mathbf{k}}\psi_0(2^{j_1}x_1 - k_1)\dots\psi_1(2^{j_i}x_i - k_i)\dots\psi_0(2^{j_d}x_d - k_d) \tag{2.14}$$

Then if we put for u:

$$u(x) = \sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^d} 4\cdot 2^{j_i} d_{\mathbf{j}\,\mathbf{k}}\psi_0(2^{j_1}x_1 - k_1)\dots\psi_0(2^{j_i}x_i - k_i)\dots\psi_0(2^{j_d}x_d - k_d) \tag{2.15}$$

We obtain:

$$u(x) = \frac{\partial v}{\partial x_i}(x) \quad \text{or, in Fourier} \quad \widehat{u}(\xi) = i\xi_i\widehat{v}(\xi)$$

Remark 2.3 *With the notation $\psi_{jk}(x) = \psi(2^j x - k)$ the relation (2.13) for $j,k \in \mathbb{Z}$ writes:*

$$i\xi\widehat{\psi_{1\,jk}} = 4\cdot 2^j\widehat{\psi_{0\,jk}}$$

This relation will prove useful in the numerical part 6.1.

3 Solving PDE's with wavelets: the Richardson iteration

Following [25, 8], we aim at solving the differential equation:

$$A\mathbf{u} = \mathbf{v} \tag{3.1}$$

where A is a linear differential operator supposed to be continuous from $H^{t/2}$ to $H^{-t/2}$ and $\mathbf{u}$ is the unknown vector function –we denote by $\mathbf{u}$ with bold character a vector function of real variables when it has several components– using the expansions of $\mathbf{u}$ and $\mathbf{v}$ in wavelet bases. We assume that we can explicitly compute the operator A but not explicitly inverse it (e.g. $A = \Delta$).

3.1 The continuous case

To solve equation (3.1), we construct an *operator* $M_\omega^\dagger$ which approximate the inverse of A:

$$M_\omega^\dagger \sim A^{-1}$$

The index ω in $M_\omega^\dagger$ means that the operator $M_\omega^\dagger$ –which will be further defined in a wavelet basis– is frequency dependent thanks to the wavelet decomposition. Then, as proposed in [8, 9], the iterative Richardson algorithm –also called *the iterative damped back-projection algorithm*– is performed by:

$$\begin{aligned} &\mathbf{u}^{(0)} = 0 \\ &\mathbf{u}^{(n)} = \mathbf{u}^{(n-1)} + M_\omega^\dagger(\mathbf{v} - A\mathbf{u}^{(n-1)}), \quad n \geq 1 \end{aligned} \tag{3.2}$$

Hence, for each $n \geq 1$, and assuming that there exists a solution $\mathbf{u}$ of (3.1),

$$\mathbf{u} - \mathbf{u}^{(n)} = (Id - M_\omega^\dagger A)(\mathbf{u} - \mathbf{u}^{(n-1)})$$

As indicated in [9], it is well known that this iterative method converges as soon as we have the *contraction property*:

$$\rho = |||Id - M_\omega^\dagger A||| < 1 \tag{3.3}$$

where Id stands for the identity operator.

3.2 The discrete formalism

Introducing a pair of biorthogonal wavelet bases $\{\psi_{jk}\}_{j,k}$ and $\{\psi^*_{jk}\}_{j,k}$ and denoting by $\underline{u} = (u_{jk})_{jk}$ the vector of wavelet coefficients: $u_{jk} =< u, \psi^*_{jk} >$ then u writes $u = \sum_{j,k} u_{jk}\psi_{jk}$ and looking for u is equivalent to looking for u_{jk}.

Let $\underline{A}$ be the variational discretization of A expressed in the wavelet basis $\{\psi_{jk}\}_{(j,k)}$ –it is called the Petrov-Galerkin stiffness matrix– $\underline{A} = (< A\psi_{j'k'}, \psi^*_{jk} >)_{j,j',k,k'}$ and $\underline{D}$ a wavelet preconditioner associated to the wavelet expansions (usually, it is the diagonal of $\underline{A}$, and has the form $\mathrm{Diag}(2^{tj})$).

We assume that in the equation $A\mathbf{u} = \mathbf{v}$, $\mathbf{u}$ and $\mathbf{v}$ are vector valued functions. For a function $\alpha : \mathcal{J} \to \mathbb{R}$ we define the ℓ^2_α norm on the wavelet coefficients $\underline{u} = (u_{\mathbf{jk}})_{\mathbf{j}\in\mathcal{J},\,\mathbf{k}\in\mathbb{Z}^d}$ by

$$\|(u_{\mathbf{jk}})_{\mathbf{j}\in\mathcal{J},\,\mathbf{k}\in\mathbb{Z}^d}\|_{\ell^2_\alpha} = \sum_{\mathbf{j}\in\mathcal{J},\,\mathbf{k}\in\mathbb{Z}^d} 2^{\alpha(\mathbf{j})}|u_{\mathbf{jk}}|^2$$

The ℓ^2_t norm corresponds either to the case $\alpha(\mathbf{j}) = tj_1$ in the MRA case $\mathbf{j} = (j_1, \varepsilon) \in \mathbb{Z} \times \{0,1\}^{d*}$, either to $\alpha(\mathbf{j}) = \max(tj_1, \ldots, tj_d)$ for the tensorial wavelets.

Let us assume that for each $\mathbf{j} \in \mathcal{J}$, $\mathbf{v_j}$ is well located in frequency domain –wavelet decompositions give us the opportunity to do this with the desired accuracy. For each $\mathbf{j} \in \mathcal{J}$, we build a matrix $M_{\omega\mathbf{j}} \in \mathbb{R}^{n\times m}$ such that:

$$M_{\omega\mathbf{j}} \approx A \qquad \text{for the frequency domain } \prod_{i=1}^{d} \pm[2^{j_i}\pi, 2^{j_i+1}\pi] \tag{3.4}$$

Then we approximate the relation $A\mathbf{u_j} = \mathbf{v_j}$ by:

$$\forall \mathbf{j} \in \mathcal{J},\ \forall \mathbf{k} \in \mathbb{Z}^d,\quad M_{\omega\,\mathbf{j}} \begin{bmatrix} u_{1\mathbf{jk}} \\ \vdots \\ u_{n\mathbf{jk}} \end{bmatrix} = \begin{bmatrix} v_{1\mathbf{jk}} \\ \vdots \\ v_{m\mathbf{jk}} \end{bmatrix} \tag{3.5}$$

Remark 3.1 *For a non-constant operator $A(x)$, we add to (3.4) a space dependency as follows:*

$$M_{\omega\,\mathbf{jk}} \approx A(\lambda) \qquad \textit{with}\ \ \lambda = 2^{-\mathbf{j}}(\mathbf{k} + (1/2, \dots, 1/2)) \tag{3.6}$$

In the frame of Richardson iteration, we take as a preconditioner $D = \sum_{\mathbf{j}\in\mathcal{J}} M_{\omega\,\mathbf{j}} Q_{\mathbf{j}}$, where $Q_{\mathbf{j}}$ is the projector on the wavelet level $W_{\mathbf{j}}$ –the space $W_{\mathbf{j}}$ is the L^2 closure of the space generated by the family $\{\mathbf{\Psi}_{\ell,\mathbf{jk}}\}_{1\le\ell\le m,\,\mathbf{k}\in\mathbb{Z}^d}$. Then the corresponding discrete preconditioner $\underline{D}$ which applies to wavelet coefficients is $\underline{D} = \sum_{\mathbf{j}\in\mathcal{J}} M_{\omega\,\mathbf{j}} \underline{Q}_{\mathbf{j}}$ where $\underline{Q}_{\mathbf{j}}$ is a diagonal matrix with ones on the lines $(\mathbf{k},\mathbf{j})_{\mathbf{k}\in\mathbb{Z}^d}$ and zeros everywhere else.
In the following, we use the notations $M_\omega = \sum_{\mathbf{j}} M_{\omega\,\mathbf{j}} Q_{\mathbf{j}}$, and $M_\omega^\dagger = \sum_{\mathbf{j}} M_{\omega\,\mathbf{j}}^\dagger Q_{\mathbf{j}}$.
If we write the sequence (3.2) with $\underline{\mathbf{v}}_n = \underline{\mathbf{v}} - \underline{A}\,\underline{\mathbf{u}}_n$, it comes:

$$\underline{\mathbf{u}}_0 = 0\ , \quad \underline{\mathbf{v}}_0 = \underline{\mathbf{v}}\ , \qquad \underline{\mathbf{u}}_{n+1} = \underline{\mathbf{u}}_n + M_\omega^\dagger \underline{\mathbf{v}}_n \quad \text{and} \quad \underline{\mathbf{v}}_{n+1} = \underline{\mathbf{v}}_n - \underline{A}(\underline{\mathbf{u}}_{n+1} - \underline{\mathbf{u}}_n) \tag{3.7}$$

Theorem 3.1 *Let $A : H^{t/2} \to H^{-t/2}$ be continuous. Let's assume that the wavelet basis $\{\mathbf{\Psi}_{\ell,\mathbf{jk}}\}_{1\le\ell\le m,\,\mathbf{j}\in\mathcal{J},\,\mathbf{k}\in\mathbb{Z}^d}$ provides a Riesz basis of $H^{\pm t/2}$ (i.e. the associated decompositions $\mathbf{v} \mapsto \underline{\mathbf{v}}$, $H^{\pm t/2} \to \ell^2_{\pm t/2}$, and reconstructions $\underline{\mathbf{v}} \mapsto \mathbf{v}$, $\ell^2_{\pm t/2} \to H^{\pm t/2}$ are continuous). Such general construction for biorthogonal wavelet bases is presented in [2].*
Moreover, we suppose we have constructed for all $\mathbf{j} \in \mathcal{J}$ matrices $M_{\omega\,\mathbf{j}} \in \mathbb{R}^{n\times m}$ such that $M_\omega^\dagger = \sum_{\mathbf{j}} M_{\omega\,\mathbf{j}}^\dagger Q_{\mathbf{j}} : H^{-t/2} \to H^{t/2}$ is continuous. We also assume that the wavelet decomposition $\mathbf{v} \mapsto (\mathbf{v_j})_{\mathbf{j}\in\mathcal{J}}$ satisfies:

$$\exists \tilde{B} > 0 \quad \textit{such that}\ \forall \mathbf{v} \in H^{-t/2}, \qquad \|(Id - A\,M_\omega^\dagger)\mathbf{v}\|^2_{H^{-t/2}} \le \tilde{B} \sum_{\mathbf{j}\in\mathcal{J}} \|(Id - A\,M_{\omega\,\mathbf{j}}^\dagger) Q_{\mathbf{j}}\mathbf{v}\|^2_{H^{-t/2}} \tag{3.8}$$

If there exists a real number $\rho \ge 0$ such that:

$$\forall \mathbf{j} \in \mathcal{J},\ \ |||(Id - A\,M_{\omega\,\mathbf{j}}^\dagger)_{|W_{\mathbf{j}}}||| \le \rho$$

i.e.

$$\forall \mathbf{j} \in \mathcal{J}, \quad \forall \mathbf{v_j} \in W_{\mathbf{j}}, \qquad \|(Id - A\,M_{\omega\,\mathbf{j}}^\dagger)\mathbf{v_j}\|_{H^{-t/2}} \le \rho \|\mathbf{v_j}\|_{H^{-t/2}} \tag{3.9}$$

then for ρ small enough, the sequence $(\underline{\mathbf{u}}_n)_{n\in\mathbb{N}}$ defined by (3.7) converges in $\ell^2_{t/2}$ to the wavelet expansion $\underline{\mathbf{u}}$ of $\mathbf{u}$ such that:

$$\underline{A}\,\underline{\mathbf{u}} = \underline{\mathbf{v}}$$

Proof: The graph of continuous operators can be summarized as follows:

$$\begin{array}{ccc} & \text{wavelet transform} & \\ \mathbf{u} \in H^{t/2} & \longleftrightarrow & \underline{\mathbf{u}} \in \ell^2_{t/2} \\ & & \\ A \downarrow \quad \uparrow M_\omega^\dagger & & \underline{M_\omega^\dagger} \uparrow \quad \downarrow \underline{A} \\ & & \\ \mathbf{v} = A\,\mathbf{u} \in H^{-t/2} & \longleftrightarrow & \underline{\mathbf{v}} = \underline{A}\,\underline{\mathbf{u}} \in \ell^2_{-t/2} \\ & \text{wavelet transform} & \end{array}$$

The operator $M_\omega^\dagger$ is not the inverse of A but its approximation.
The choice of $\underline{A}$ as the Petrov-Galerkin stiffness matrix: $a_{\ell\ell'\mathbf{jj'kk'}} = \langle A\Psi_{\ell',\mathbf{j'k'}}, \Psi^*_{\ell,\mathbf{jk}}\rangle$ infers

$$A\mathbf{u} = \sum_{\ell,\mathbf{jk}} \langle A\mathbf{u}, \Psi^*_{\ell,\mathbf{jk}}\rangle \Psi_{\ell,\mathbf{jk}} = \sum_{\ell,\mathbf{jk}} \langle A \sum_{\ell',\mathbf{j'k'}} u_{\ell',\mathbf{j'k'}} \Psi_{\ell',\mathbf{j'k'}}, \Psi^*_{\ell,\mathbf{jk}}\rangle \Psi_{\ell,\mathbf{jk}} = \sum_{\ell,\mathbf{jk}} (\underline{A}\,\underline{\mathbf{u}})_{\ell,\mathbf{jk}} \Psi_{\ell,\mathbf{jk}}$$

hence $\underline{A}\,\underline{\mathbf{u}} = \underline{\mathbf{v}}$ iff $A\mathbf{u} = \mathbf{v}$.

As the wavelet decomposition is continuous on $H^{-t/2}$,

$$\exists b, B > 0 \quad \text{such that} \quad \forall \mathbf{v} \in H^{-t/2}, \qquad b \sum_{\mathbf{j}\in\mathcal{J}} \|\mathbf{v_j}\|^2_{H^{-t/2}} \le \|\mathbf{v}\|^2_{H^{-t/2}} \le B \sum_{\mathbf{j}\in\mathcal{J}} \|\mathbf{v_j}\|^2_{H^{-t/2}} \tag{3.10}$$

Properties (3.8), (3.9) and (3.10) imply:

$$\|\mathbf{v}_{n+1}\|^2_{H^{-t/2}} \le \tilde{B} \sum_{\mathbf{j}\in\mathcal{J}} \|(Id - A\,M_\omega^\dagger)\mathbf{v}_{n\,\mathbf{j}}\|^2_{H^{-t/2}} \le \tilde{B} \sum_{\mathbf{j}\in\mathcal{J}} \rho^2 \|\mathbf{v}_{n\,\mathbf{j}}\|^2_{H^{-t/2}} \le \rho^2 \frac{\tilde{B}}{b} \|\mathbf{v}_n\|^2_{H^{-t/2}}$$

If $\rho^2 \tilde{B}/b < 1$, as $M_\omega^\dagger$ is continuous, the series $\sum_n M_\omega^\dagger \mathbf{v}_n$ converges in the Banach space $H^{t/2}$ to a solution $\mathbf{u}$ of the equation $A\mathbf{u} = \mathbf{v}$.
Thanks to the isomorphism between the continuous case and the discretized case with infinite expansion, this is equivalent to $\underline{A}\,\underline{\mathbf{u}} = \underline{\mathbf{v}}$. If we truncate the wavelet expansion, then there are non-linear approximation issues. These are addressed in reference [4] for instance. ■

The ideal wavelets that provide a minimal compact support for the Fourier transform are the Shannon wavelets. As the compact supports of the Fourier transforms of Shannon wavelets from different levels are almost disjoint, we have $\tilde{B} = b = 1$ for all A linear operator with constant coefficients. Shannon wavelets have an infinite support and are not used in practice. But, in first approximation, they can serve as a model –see part 2.2.

4 Symbols of operators constructed thanks to wavelet decompositions

4.1 Symbol of an operator

The convergence of the Richardson iteration in (3.2) involves an approximation operator M_ω relying on a partition of the spectrum of the operator A. In our case, wavelet decomposition provides this partition. Hence we need the notion of symbol as introduced by Lars Hörmander in his book *The Analysis of Partial Differential Operators* [18], applied to wavelets.

We denote by ∂_j the differentiation along the variable x_j, and by D_j the differentiation $-i\partial_j$. For $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, we write $D^\alpha = D_1^{\alpha_1} \dots D_d^{\alpha_d}$. Let u denote a Schwartz function of d real variables (i.e. u is C^∞ and fast decreasing: $\forall N \in \mathbb{N}$, $\exists B > 0$ s.t. $\forall x \in \mathbb{R}^d$, $|u(x)| < B/(1 + |x|^2)^{N/2}$). We denote by $\langle \cdot, \cdot \rangle$ the scalar product either on vectors either in dual spaces.

We recall that $\widehat{u}$ stands for the Fourier transform of u, i.e.

$$\widehat{u}(\xi) = \int_{\xi\in\mathbb{R}^d} e^{-i\langle x,\xi\rangle} u(x) dx$$

We also denote by $\mathcal{F}$ the isomorphism of $L^2(\mathbb{R}^d)$ given by $\mathcal{F} : u \mapsto (2\pi)^{-d/2}\widehat{u}$.

The inverse Fourier transform is done by:

$$u(x) = (2\pi)^{-d/2} \int_{\xi\in\mathbb{R}^d} e^{i\langle x,\xi\rangle} \mathcal{F}u(\xi)d\xi \tag{4.1}$$

When we differentiate the relation (4.1), it yields:

$$D^\alpha u(x) = (2\pi)^{-d/2} \int_{\xi\in\mathbb{R}^d} e^{i\langle x,\xi\rangle} \xi^\alpha \mathcal{F}u(\xi)d\xi$$

Thus differentiating u by D^α consists in multiplying the Fourier transform of u by $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_d^{\alpha_d}$. The function $\xi \mapsto \xi^\alpha$ is called the symbol of D^α. More generally speaking, if $a(\xi)$ is a C^∞ function slowly increasing (i.e. such that $\exists N \in \mathbb{N},\ B > 0$ s.t. $\forall \xi \in \mathbb{R}^d,\ |a(\xi)| < B(1 + |\xi|^2)^{N/2}$), $a(D)$ defines an operator of symbol $a(\xi)$ acting on the class of the Schwartz functions $\mathcal{S}$ by

$$a(D)u(x) = (2\pi)^{-d/2} \int_{\xi\in\mathbb{R}^d} e^{i\langle x,\xi\rangle} a(\xi)\mathcal{F}u(\xi)d\xi \tag{4.2}$$

Let's now consider a differential operator A of order m with variable coefficients a_α in $\mathcal{S}$, $A = \sum_{|\alpha|=0}^m a_\alpha(x)D^\alpha$. Then, instead of considering the formula:

$$Au = (2\pi)^{-d/2} \int_{\xi\in\mathbb{R}^d} e^{i\langle x,\xi\rangle} \mathcal{F}(Au)(\xi)d\xi$$

where

$$\mathcal{F}(Au)(\xi) = (2\pi)^{d/2} \sum_{|\alpha|=0}^{m} (\mathcal{F}a_\alpha) * (\xi^\alpha \mathcal{F}u)$$

which is not a multiplication but an integral operator on $\mathcal{F}u$, we consider the formula (for $x \in \mathbb{R}^d$):

$$Au(x) = (2\pi)^{-d/2} \int_{\xi\in\mathbb{R}^d} e^{i\langle x,\xi\rangle} \left(\sum_{|\alpha|=0}^{m} a_\alpha(x)\xi^\alpha \right) \mathcal{F}(u)(\xi)d\xi$$

We write it:

$$Au(x) = (2\pi)^{-d/2} \int_{\xi\in\mathbb{R}^d} e^{i\langle x,\xi\rangle} p(x,\xi)\mathcal{F}(u)(\xi)d\xi \tag{4.3}$$

introducing the *symbol* $p(x, \xi)$ of P

$$p(x,\xi) = \sum_{|\alpha|=0}^{m} a_\alpha(x)\xi^\alpha$$

The formula (4.3) gives us the possibility to define the operators $p(x, D)$ with symbols $p(x, \xi)$ which are not polynomials in ξ. These operators are called pseudo-differential. The functions p must verify regularity and growth properties of polynomial type (see [18]).

We'll remark that the function $\mathcal{F}(Pu)(\xi)$ is no more the function $p(x, \xi)\mathcal{F}u(\xi)$ which appears in (4.3) since the latter depends on x.

In the following, we will need differential operators applied to vector functions $\mathbb{R}^d \to \mathbb{R}^n$. For $\mathbf{u}$ having several components, the symbol $p(x,\xi)$ is a matrix: $\forall x, \xi \in \mathbb{R}^d$, $M(x,\xi) \in \mathbb{C}^{n\times n}$. Let $A = (A_{ij})_{1\le i\le n, 1\le j\le n}$ be a differential operator $A : \left(H^s(\mathbb{R}^d)\right)^n \to \left(H^r(\mathbb{R}^d)\right)^n$, with:

$$A_{ij} = \sum_\alpha a_{ij,\alpha}(x) D^\alpha$$

Its symbol is $M = (m_{ij})_{1\le i\le n, 1\le j\le m}$ with

$$m_{ij}(x,\xi) = \sum_\alpha a_{ij,\alpha}(x)\xi^\alpha$$

We apply the operator A componentwise as follows:

$$(A\mathbf{u})_i = \sum_{j=1}^m m_{ij}(x,D)u_j$$

Therefore, the multidimensional symbol can be handled similarly as in 1-D.

Remark 4.1 *As the operator is applied to real functional spaces, its symbol verifies the same relation as the Fourier transform of real functions, that is:*

$$\forall i,j \qquad m_{ij}(x,-\xi) = \overline{m_{ij}(x,\xi)}$$

where $\overline{z}$ denotes the complex conjugate of z.

From now on we think of A as being an operator with constant coefficients and $n \le 1$ being a natural number. If we denote by $M(\xi)$ the symbol associated to A, we express A after a Fourier transform of the equation (3.1):

$$M(\xi)\widehat{\mathbf{u}} = \widehat{\mathbf{v}} \qquad \text{and the pseudo-inverse solution} \qquad \widehat{\mathbf{u}} = M(\xi)^\dagger \widehat{\mathbf{v}}$$

with $\mathbf{u} \in (H^{t/2}(\mathbb{R}^d))^n$, $\mathbf{v} \in (H^{-t/2}(\mathbb{R}^d))^n$, $M(\xi) \in \mathbb{C}^{n\times n}(\mathbb{R}^d)$ and $M(\xi)^\dagger$ the pseudo-inverse of $M(\xi)$.

Wavelet and symbol

Proposition 4.1 *If we consider two wavelets ψ_0 and ψ_1, then the operator*

$$A : \left\{ \begin{array}{rcl} W_{0\,j} & \to & W_{1\,j} \\ u_j = \sum_{k\in\mathbb{Z}} d_{jk}\psi_{0\,jk} & \mapsto & Au_j = \sum_{k\in\mathbb{Z}} d_{jk}\psi_{1\,jk} \end{array} \right.$$

where $W_{i\,j} = span\{\psi_i(2^j x - k), k \in \mathbb{Z}\}$, corresponds, for the Fourier transforms to the operation:

$$\widehat{Au_j}(\xi) = \frac{\widehat{\psi_1}(2^{-j}\xi)}{\widehat{\psi_0}(2^{-j}\xi)}\widehat{u_j}(\xi)$$

Hence the symbol of A writes: $p(\xi) = \frac{\widehat{\psi_1}(2^{-j}\xi)}{\widehat{\psi_0}(2^{-j}\xi)}$

In order to help the reader checking this assertion, we recall that for $u \in L^2(\mathbb{R})$, $a \neq 0$ and $b \in \mathbb{R}$,

$$\widehat{u(a\cdot - b)}(\xi) = \frac{1}{a} e^{-i\frac{b}{a}\xi} \widehat{u}\left(\frac{\xi}{a}\right)$$

4.2 Constructible operators

Here we restrict ourselves to the case $m = n$. From the results of the previous sections, we infer:

Theorem 4.1 (Set of constructible operators) *Consider the set $\Upsilon_{\mathbf{j}}$ of all the operators obtained by:*

$$M_{\omega\mathbf{j}} :$$

$$\begin{array}{ccccc} & (W_{\mathbf{j}})^n & \to & \left(H^t(\mathbb{R}^d)\right)^n & \\ u_{\mathbf{j}} = & \left| \begin{array}{l} \sum_{k\in\mathbb{Z}^d} d_{jk}^{(1)} \psi_{0\,j_1k_1} \times \cdots \times \psi_{0\,j_dk_d} \\ \vdots \\ \sum_{k\in\mathbb{Z}^d} d_{jk}^{(n)} \psi_{0\,j_1k_1} \times \cdots \times \psi_{0\,j_dk_d} \end{array} \right. & \mapsto & M_{\omega\mathbf{j}}u_{\mathbf{j}} = & \left| \begin{array}{l} \sum_{\eta\in\mathcal{H}} \sum_{k\in\mathbb{Z}^d} \left(\sum_{i_2=1}^n a_{\eta 1 i_2} d_{jk}^{(i_2)}\right) \psi_{\eta_1\,j_1k_1} \times \cdots \times \psi_{\eta_d\,j_dk_d} \\ \vdots \\ \sum_{\eta\in\mathcal{H}} \sum_{k\in\mathbb{Z}^d} \left(\sum_{i_2=1}^n a_{\eta n i_2} d_{jk}^{(i_2)}\right) \psi_{\eta_1\,j_1k_1} \times \cdots \times \psi_{\eta_d\,j_dk_d} \end{array} \right. \end{array} \tag{4.4}$$

with $\mathcal{H} \subset \mathbb{Z}^d$ a finite set, and where we noted $\psi'_{k+1} = 4\,\psi_k$ for $k \in \mathbb{Z}$. Then the set of symbols of $\Upsilon_{\mathbf{j}}$ is equal to the $\mathbb{R}$-algebra of $\{M : \mathbb{R} \to \mathbb{C}^{n\times n},\ \xi \mapsto M(\xi),\ +\,*\}$ generated by the constants $\mathcal{M}_n(\mathbb{R})$ and the elements $\{i\xi_\ell I\}_{1\le\ell\le d}$ and $\{i\xi_\ell^{-1} I\}_{1\le\ell\le d}$.

Proof: Let $M_{\omega\mathbf{j}}^{(1)}$ and $M_{\omega\mathbf{j}}^{(2)}$ be two elements of $\Upsilon_{\mathbf{j}}$.

- Then $M_{\omega\mathbf{j}} = M_{\omega\mathbf{j}}^{(1)} + M_{\omega\mathbf{j}}^{(2)}$ with $\mathcal{H} = \mathcal{H}^{(1)} \cup \mathcal{H}^{(2)}$ and $a_{\eta i_1 i_2} = a_{\eta i_1 i_2}^{(1)} + a_{\eta i_1 i_2}^{(2)}$ is an element of $\Upsilon_{\mathbf{j}}$,
- as well as $M_{\omega\mathbf{j}} = M_{\omega\mathbf{j}}^{(1)} \circ M_{\omega\mathbf{j}}^{(2)}$ with $\mathcal{H} = \mathcal{H}^{(1)} + \mathcal{H}^{(2)} = \{\eta^{(1)} + \eta^{(2)}, \eta^{(1)} \in \mathcal{H}^{(1)}, \eta^{(2)} \in \mathcal{H}^{(2)}\}$ and $M_{\omega\mathbf{j}} = \sum_{\eta\in\mathcal{H}} M_{\omega\mathbf{j}\eta}$ with, if we denote by $A_\eta = (a_{\eta i_1 i_2})$, $A_\eta^{(1)} = (a_{\eta i_1 i_2}^{(1)})$ and $A_\eta^{(2)} = (a_{\eta i_1 i_2}^{(2)})$ the matrices of coefficients respectively associated to $M_{\omega\mathbf{j}\eta}$, $M_{\omega\mathbf{j}\eta}^{(1)}$ and $M_{\omega\mathbf{j}\eta}^{(2)}$,
$$A_\eta = \sum_{\eta^{(1)}+\eta^{(2)}=\eta} A_{\eta^{(1)}}^{(1)} A_{\eta^{(2)}}^{(2)}$$

Then the set $\Upsilon_{\mathbf{j}}$ is stable under the operations $+$ and $\circ$ which correspond to the operations $+$ and $*$ for the symbol matrices.
On the other hand, given $M = (m_{i_1i_2}) \in \mathcal{M}_n(\mathbb{R})$, the element $M_{\omega\mathbf{j}}$ of $\Upsilon_{\mathbf{j}}$ with $\mathcal{H} = \{(0,\ldots,0)\}$ and $a_{0 i_1 i_2} = m_{i_1i_2}$ provides the constant M. The elements $\{i\xi_\ell I\}$ and $\{i\xi_\ell^{-1} I\}$ are constructed thanks to operations (2.14) and (2.15) on the wavelet basis and wavelet coefficients.
Conversely, all symbols of operators $M_{\omega\mathbf{j}} \in \Upsilon_{\mathbf{j}}$ are polynomials of $i\xi_\ell$ and $(i\xi_\ell)^{-1}$ and therefore generated by $\mathcal{M}_n(\mathbb{R})$, $\{i\xi_\ell I\}_{1\le\ell\le d}$ and $\{i\xi_\ell^{-1} I\}_{1\le\ell\le d}$. ■

This theorem enables us to diversify the wavelet approximations of differential operators. It extends the result of section 3. But it still remains rather limited since, for instance, in dimension larger than 2, the inverse Laplace operator Δ^{-1} doesn't belong to $\Upsilon_{\mathbf{j}}$ as constructed in theorem 4.1.

This kind of operator construction is presented in [1] but without the differentiation and in the frame of Bessel multipliers for an other purpose than operator approximation.

Remark 4.2 *Nevertheless, thanks to Weierstraß theorem, given any operator A with symbol $M(\xi_1^2, \ldots, \xi_d^2)$ and any wavelet ψ_0 whose Fourier transform has a compact support included in $\mathbb{R}^*$ (such as Meyer wavelets), this theorem allows us to construct operators which converge uniformly to A on $(W_{\mathbf{j}})^n$.*

Remark 4.3 *The operation (4.4) is equivalent to*

$$
\begin{aligned}
M_{\omega\mathbf{j}} :&\\
&(W_{\mathbf{j}})^n \to \left(H^t(\mathbb{R}^d)\right)^n\\
u_{\mathbf{j}} =& \left| \begin{array}{l} \sum_{\eta\in\mathcal{H}} \sum_{k\in\mathbb{Z}^d} d_{jk}^{(1,\eta)} \psi_{-\eta_1\, j_1 k_1} \times \cdots \times \psi_{-\eta_d\, j_d k_d} \\ \vdots \\ \sum_{\eta\in\mathcal{H}} \sum_{k\in\mathbb{Z}^d} d_{jk}^{(d,\eta)} \psi_{-\eta_1\, j_1 k_1} \times \cdots \times \psi_{-\eta_d\, j_d k_d} \end{array} \right. \qquad \text{for all} \quad \eta \in \mathcal{H}\\
&\mapsto M_{\omega\mathbf{j}} u_{\mathbf{j}} = \left| \begin{array}{l} \sum_{k\in\mathbb{Z}^d} \sum_{\eta\in\mathcal{H}} \left(\sum_{i_2=1}^n a_{\eta 1 i_2} d_{jk}^{(i_2,\eta)} \right) \psi_{0\, j_1 k_1} \times \cdots \times \psi_{0\, j_d k_d} \\ \vdots \\ \sum_{k\in\mathbb{Z}^d} \sum_{\eta\in\mathcal{H}} \left(\sum_{i_2=1}^n a_{\eta n i_2} d_{jk}^{(i_2,\eta)} \right) \psi_{0\, j_1 k_1} \times \cdots \times \psi_{0\, j_d k_d} \end{array} \right.
\end{aligned}
\tag{4.5}
$$

It corresponds to decompose $u_{\mathbf{j}}$ in various wavelet bases indexed by $-\eta$. And then reconstruct the result of the operator $M_{\omega\mathbf{j}}$ in the wavelet basis $(\psi_{0\, j_1 k_1} \times \cdots \times \psi_{0\, j_d k_d})_{\mathbf{k}} \otimes \cdots \otimes (\psi_{0\, j_1 k_1} \times \cdots \times \psi_{0\, j_d k_d})_{\mathbf{k}}$. Then it is more convenient to apply the operator A on $\mathbf{u}$ when it is expanded in only one wavelet basis. In the numerical experiments, part 6, we use this latest form.

5 Convergence criteria for the Richardson iteration

5.1 Multiresolution analysis (MRA) versus tensorial basis

We consider two different wavelet decompositions for a function on $\mathbb{R}^d$ with $d \geq 2$: the multidimensional multiresolution analysis decomposition and the tensorial wavelet decomposition. The MRA decomposition of a function u in 2D writes:

$$
\begin{aligned}
u(x_1, x_2) \;=\; \sum_{j\in\mathbb{Z}} \Bigg(& \sum_{(k_1,k_2)\in\mathbb{Z}^2} u_{j,k_1,k_2}^{(1,0)} \; \psi_{0\, j,k_1}(x_1)\; \varphi_{1\, j,k_2}(x_2) + \sum_{(k_1,k_2)\in\mathbb{Z}^2} u_{j,k_1,k_2}^{(0,1)} \; \varphi_{0\, j,k_1}(x_1)\; \psi_{1\, j,k_2}(x_2) \\
& + \sum_{(k_1,k_2)\in\mathbb{Z}^2} u_{j,k_1,k_2}^{(1,1)} \; \psi_{0\, j,k_1}(x_1)\; \psi_{1\, j,k_2}(x_2) \Bigg)
\end{aligned}
$$

with the notation $\psi_{j,k}(x) = \psi(2^j\, x - k)$.
While the tensorial wavelet decomposition writes:

$$
u(x_1, x_2) \;=\; \sum_{(j_1,j_2)\in\mathbb{Z}^2} \sum_{(k_1,k_2)\in\mathbb{Z}^2} u_{j_1,j_2,k_1,k_2} \; \psi_0(2^{j_1} x_1 - k_1)\; \psi_1(2^{j_2} x_2 - k_2)
$$

These two decompositions correspond to two different partitions of the Fourier space (i.e. frequency domain). These are represented in figures 2 and 3. On each figure, in the last square, which corresponds to the wavelet transform, the lower frequencies are localized in the upper left corner of the square of coefficients, and the higher frequencies in the bottom right side.

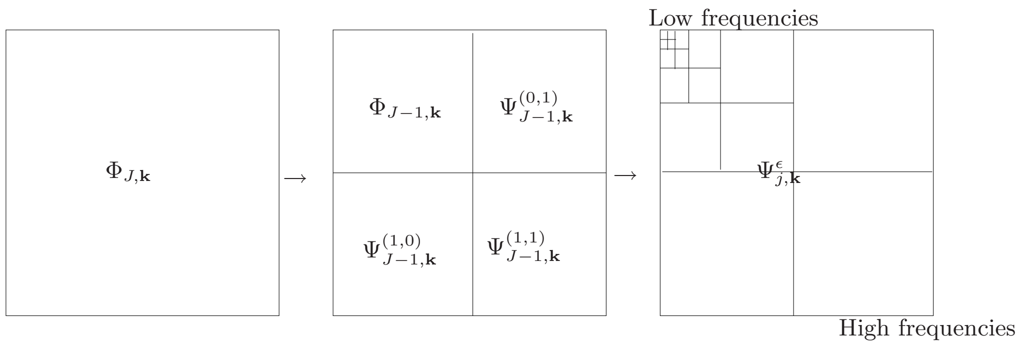

Figure 2: Splitting of the Fourier modes induced by the 2D-MRA wavelet decomposition

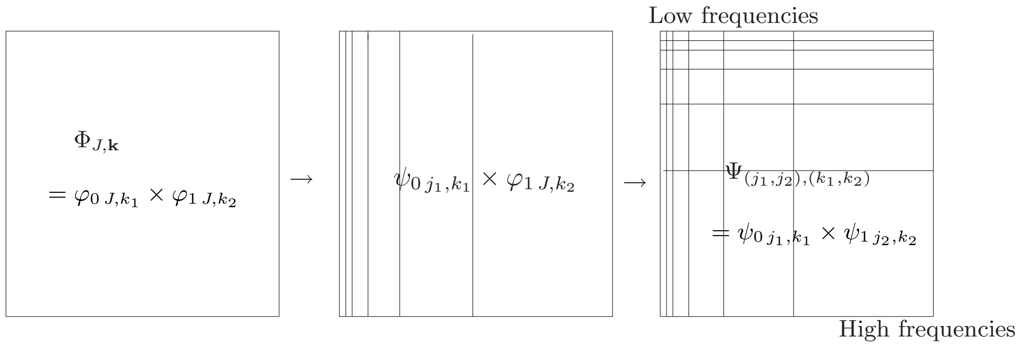

Figure 3: Fourier splitting induced by the tensorial wavelet decomposition

5.2 Convergence theorems with Shannon wavelets

To begin with, we'll only consider approximation matrices that are constant over each frequency domain indexed by $\mathbf{j} \in \mathbb{Z}^d$.

The two decompositions, MRA and tensorial, induce different conditions for the approximation of the symbol matrix $M(\xi)$. Following part 3, MRA and tensorial wavelet convergence theorems result as follows:

Theorem 5.1 (MRA) *If the symbol matrix $M(\xi)$ admits a pseudo-inverse $M(\xi)^\dagger$ such that $M(\xi)M(\xi)^\dagger = Id\ \forall\xi \neq (0,\dots,0)$, and if $\exists\rho < 1$ such that $\forall j \in \mathbb{Z}$ and $\forall\varepsilon \in \{0,1\}^d \setminus \{(0,\dots,0)\}\ \exists M_{\omega\,(j,\varepsilon)} \in \mathbb{R}^{n\times m}$ such that $\forall\xi \in \prod_{i=1}^d \pm[\varepsilon_i 2^j\pi, (\varepsilon_i+1)2^j\pi]$, $\|Id - M(\xi)M^\dagger_{\omega\,(j,\varepsilon)}\| \leq \rho$ then the sequence (3.2) using the MRA decomposition with Shannon wavelets, converges to the solution of the PDE.*

Proof: We recall that we are in the case $b = \tilde{B} = 1$ of theorem 3.1 since we deal with Shannon wavelets.
The partition of the support of $\widehat{\mathbf{v}}$ operated by the MRA decomposition is the following:

$$\mathcal{J} = \{(j,\varepsilon) \in \mathbb{Z} \times \{0,1\}^{d,*}\}$$

$$\mathbf{v} = \sum_{(j,\varepsilon)\in\mathcal{J}} \mathbf{v}_(j,\varepsilon) \quad \text{with} \quad supp\,\widehat{\mathbf{v}}_{(j,\varepsilon)} \subset \prod_{i=1}^{d} \pm[\varepsilon_i 2^j\pi, (\varepsilon_i+1)2^j\pi]$$

In the case $\varepsilon_i = 0$, a scaling function φ_j appears for the variable x_i, while in the case $\varepsilon_i = 1$ this is a wavelet function ψ_j. Owing to the fact that $\operatorname{supp}\widehat{\varphi}_j \subset [-2^j\pi, 2^j\pi]$ and $\operatorname{supp}\widehat{\psi}_j \subset [-2^{j+1}\pi, -2^j\pi] \cup [2^j\pi, 2^{j+1}\pi]$, we obtain the sets indicated in the theorem. This case is represented in Figure 2.
Then we apply theorem 3.1 to obtain the convergence. ■

Remark 5.1 *The fact that $M(\xi)$ admits a pseudo-inverse $M(\xi)^\dagger$ such that $M(\xi)M(\xi)^\dagger = Id$ is implied by $\exists \rho < 1,\ \exists M_{\omega\mathbf{j}} \in \mathbb{R}^{n\times m}$ such that $\|Id - M(\xi)M_{\omega\mathbf{j}}^\dagger\| \le \rho$.*

Theorem 5.2 (Tensorial wavelets) *If the symbol matrix $M(\xi)$ satisfies $\exists \rho < 1$ such that $\forall \mathbf{j} \in \mathbb{Z}^d\ \ \exists M_{\omega\mathbf{j}} \in \mathbb{R}^{n\times m}$ such that $\forall \xi \in \prod_{i=1}^d \pm[2^{j_i}\pi, 2^{j_i+1}\pi],\ \ \|Id - M(\xi)M_{\omega\mathbf{j}}^\dagger\| \le \rho$ then the method converges with the tensorial wavelet decomposition.*

Proof: Anew, we use theorem 3.1 with $\mathcal{J} = \mathbb{Z}^d$ and

$$\mathbf{v} = \sum_{\mathbf{j}\in\mathcal{J}} \mathbf{v_j} \quad \text{with} \quad supp\, \widehat{\mathbf{v}}_\mathbf{j} \subset \pm[2^{j_i}\pi, 2^{j_i+1}\pi]$$

This wavelet decomposition parts the frequency domain as represented in Figure 3. ■

Remark 5.2 *If we consider only constant matrices operating on the wavelet coefficients, the resulting operations on the Fourier transform of the functions are symmetric by reflection along all axes:*

$$\forall i \in \{1,\dots,d\},\ M_{\omega\mathbf{j}}(\xi_1,\dots,-\xi_i,\dots,\xi_d) = M_{\omega\mathbf{j}}(\xi_1,\dots,\xi_i,\dots,\xi_d) \tag{5.1}$$

On the other hand, as we deal with real functions, the approximation matrix must be real.

Remark 5.3 *The best approximation $M_{\omega\mathbf{j}}$ of $M(\xi)$, for the inversion is given by*

$$M_{\omega\mathbf{j}} = \arg\min_{\mu\in\mathbb{R}^{n\times m}} \sup_{\xi\in supp(\widehat{\mathbf{v}}_\mathbf{j})} \|Id - M(\xi)\mu^\dagger\|$$

Example 5.1 *The operator Δ^{-1} matches the two cases. Richardson iteration converges for MRA wavelets and for tensorial wavelets.*

Example 5.2 *If we consider the 1-D symbols $p : \mathbb{R} \to \mathbb{C},\ \xi \mapsto p(\xi)$ that are continuous, an example that doesn't match the conditions of the theorem is given by the symbol $p(\xi) = e^{i\xi}$ of the operator $P : u \mapsto u(\cdot + 1)$. The symbol p doesn't satisfy the condition $\exists \rho < 1,\ \exists \mu \in \mathbb{R}$ such that $\forall \xi \in [\pi, 2\pi],\ \ \|1 - p(\xi)\mu^{-1}\| \le \rho$.*

Example 5.3 *The 1-D symbols $p : \mathbb{R} \to \mathbb{R},\ \xi \mapsto p(\xi)$ that are continuous, and verify $\forall \xi \neq 0,\ p(\xi) \neq 0$ and*

$$\sup_{j\in\mathbb{Z}} \frac{\sup_{\xi\in[2^j\pi, 2^{j+1}\pi]}(|p(\xi)|)}{\inf_{\xi\in[2^j\pi, 2^{j+1}\pi]}(|p(\xi)|)} < +\infty$$

can be approximated by a constant ω_j on each interval $\pm[2^j\pi, 2^{j+1}\pi]$ with optimal value verifying

$$\omega_j^{-1} = \frac{1}{2}(a_j^{-1} + b_j^{-1})$$

with $a_j = \inf_{\xi\in[2^j\pi, 2^{j+1}\pi]}(p(\xi))$ and $b_j = \sup_{\xi\in[2^j\pi, 2^{j+1}\pi]}(p(\xi))$
That is the case for symbols with polynomial increase, since for $p(\xi) = \xi^\alpha$, $\frac{(2^{j+1}\pi)^\alpha}{(2^j\pi)^\alpha} = 2^\alpha$.

Example 5.4 *In 2-D, even for real operator matrices, the approximation by constant matrices can fail. For instance, if we consider a symbol matrix M such that:*

$$M(\xi_1,\xi_2) = \begin{pmatrix} \cos(\xi_1) & -\sin(\xi_1) \\ \sin(\xi_1) & \cos(\xi_1) \end{pmatrix} \qquad \text{for } \ \xi \in [\pi, 2\pi]^2$$

any wavelet approximation by constant matrices $\mu \in \mathbb{R}^{2\times 2}$ fails: either $\|Id+\mu\| \geq 1$ or $\|Id-\mu\| \geq 1$.

5.3 Convergence with the wavelet packets

Let A be an operator from $(H^{t/2}(\mathbb{R}^d))^n$ to $(H^{-t/2}(\mathbb{R}^d))^n$, with symbol $M(\xi)$. Assume that $M(\xi)$ is continuous and invertible almost everywhere on $\mathbb{R}^d$ in the sense of the Riemann measure (i.e. such that for all compact sets K of $\mathbb{R}^d$, $K \cap (\det(M))^{-1}(\{0\})$ the subset of K where $M(\xi)$ is not invertible has a vanishing Riemann measure), and verifies the condition (5.1). Then it can be approximated by constant matrices $M_{\omega\,\mathbf{j}}$ with Shannon wavelet packets providing an *ad hoc* partition of the frequency domain.

Theorem 5.3 *For all linear operators A with constant coefficients satisfying the above conditions, we can numerically solve the equation $A\mathbf{u} = \mathbf{v}$ on $\mathbb{R}^d$ with a wavelet packet method: $\forall \varepsilon > 0$, we can find $\mathbf{u}_\varepsilon$ such that $\|\mathbf{v} - A\mathbf{u}_\epsilon\| < \varepsilon$, thanks to the wavelet algorithms described in part 3.*

Proof: First we build a finite set of rectangles $\{\omega_\mathbf{j}\}_{\mathbf{j}\in\mathcal{J}}$ such that

$$\omega_\mathbf{j} = \prod_{i=1}^{d} \left[2^{j_i}\ell_i, 2^{j_i}(\ell_i+1)\right]$$

with $\mathbf{j} = (j_1,\ldots,j_d,\ell_1,\ldots,\ell_d)$, $j_i \in \mathbb{Z}$ and $\ell_i \in \mathbb{N}$. The sets $\{\omega_\mathbf{j}\}_{\mathbf{j}\in\mathcal{J}}$ correspond to compact supports of Fourier transforms of the Shannon wavelet packets $\widehat{\Psi_\varepsilon}(\xi)$.

Thanks to the invertibility of $M(\xi)$ almost everywhere in $\mathbb{R}^d$ for Riemann measure, we construct $\{\omega_\mathbf{j}\}_{\mathbf{j}\in\mathcal{J}}$ such that for $\Omega = \bigcup_{\mathbf{j}\in\mathcal{J}} \omega_\mathbf{j}$, $\|\widehat{v} - \widehat{v}_{|\Omega}\| < \varepsilon/2$, and for $\xi_\mathbf{j} = (2^{j_1}\ell_1, \ldots, 2^{j_i}\ell_i, \ldots, 2^{j_d}\ell_d) \in \omega_\mathbf{j}$,

$$\sup_{\mathbf{j}} \left(\sup_{\xi\in\omega_\mathbf{j}} \|Id - M(\xi)M^{-1}(\xi_\mathbf{j})\| \right) < 1$$

Remark that, as a consequence, $M(\xi)$ is invertible on Ω.
On the set Ω, we can apply the wavelet algorithm of part 3 with Shannon wavelet packets and $M_\omega^\dagger = \sum_{\mathbf{j}\in\mathcal{J}} M^{-1}(\xi_\mathbf{j}) Q_\mathbf{j}$, and get $\mathbf{u}_\epsilon$ such that $\|\widehat{\mathbf{v}}_{|\Omega} - M_{|\Omega}(\xi)\widehat{\mathbf{u}}_{\epsilon|\Omega}\| < \varepsilon/2$, then we extend $\widehat{\mathbf{u}_\epsilon}$ to $\mathbb{R}^d$ by taking $\widehat{\mathbf{u}}_{\epsilon|^c\Omega} = 0$.

Remark 5.4 *This approach is valid for Shannon wavelets. But in practice we would like to use other wavelets for which the frequency partition induced by the wavelet packets is very difficult to control.*

5.4 Convergence in the general case

In order to use general wavelets, we have to express the wavelet projectors in terms of Fourier transform. Let Q_j be the biorthogonal projector onto a 1D wavelet space $W_j = span\{\psi_{jk}\}_{k\in\mathbb{Z}}$. Then, as indicated in [20], the wavelet level j of a function u:

$$Q_j u(x) = \sum_{k\in\mathbb{Z}} 2^j \langle u, \psi^*(2^j x - k)\rangle \psi(2^j x - k) \tag{5.2}$$

writes in Fourier domain:

$$\widehat{Q_j u}(\xi) = \left[\sum_{k\in\mathbb{Z}} \widehat{u}(\xi + 2k\pi 2^j)\overline{\widehat{\psi^*}}\left(\frac{\xi}{2^j} + 2k\pi\right)\right] \widehat{\psi}\left(\frac{\xi}{2^j}\right) \tag{5.3}$$

where parameter k doesn't refer to any space localization. It is a frequency shift. The series (5.3) can be written: $\widehat{Q_j u}(\xi) = \mu(\xi)\widehat{\psi}\left(\frac{\xi}{2^j}\right)$, with μ a $2\pi 2^j$ periodic function. With Shannon wavelets, it reads $\widehat{Q_j u}(\xi) = \chi_{\pm 2^j[\pi,2\pi]}(\xi)\widehat{u}(\xi)$.

A similar result applies to the projection P_j on $V_j = span(\varphi_{jk})_{k\in\mathbb{Z}}$ replacing ψ^* by φ^* and ψ by φ in the series (5.2). It extends naturally to $\mathbb{R}^d$ with the tensor product.

To prove the convergence of $(\mathbf{u}^{(p)})$ to the solution $\mathbf{u}$ in the Richardson iteration (3.2), we consider the operator $(Id - M_\omega^\dagger A)$ instead of $(Id - AM_\omega^\dagger)$, and $m = n$. We express the operator $(Id - M_\omega^\dagger A)$ with general wavelets in the Fourier domain:

$$\begin{aligned}
\widehat{(Id - M_\omega^\dagger A)\mathbf{u}}(\xi) &= \widehat{\mathbf{u}}(\xi) - \sum_{\mathbf{j}\in\mathbb{Z}^d} \widehat{M_{\omega\,\mathbf{j}}^\dagger Q_\mathbf{j} A\,\mathbf{u}}(\xi) \qquad (5.4)\\
&= \widehat{\mathbf{u}}(\xi) - \sum_{\mathbf{j}\in\mathbb{Z}^d} M_{\omega\,\mathbf{j}}^\dagger \left[\sum_{\mathbf{k}\in\mathbb{Z}^d} \left(M(\xi + 2\mathbf{k}\pi 2^\mathbf{j})\widehat{\mathbf{u}}(\xi + 2\mathbf{k}\pi 2^\mathbf{j})\right)\overline{\widehat{\Psi^*}}\left(\frac{\xi}{2^\mathbf{j}} + 2\mathbf{k}\pi 2^\mathbf{j}\right)\right]\widehat{\Psi}\left(\frac{\xi}{2^\mathbf{j}}\right)
\end{aligned}$$

with the notations $\xi + 2\mathbf{k}\pi 2^\mathbf{j} = \left(\xi_1 + 2k_1\pi 2^{j_1}, \ldots, \xi_d + 2k_d\pi 2^{j_d}\right)$, and

$$\widehat{\mathbf{u}}(\xi)\widehat{\Psi(\xi)} = \begin{bmatrix} \widehat{u_1}(\xi)\widehat{\psi(\xi_1)}\ldots\widehat{\psi(\xi_d)} \\ \vdots \\ \widehat{u_n}(\xi)\widehat{\psi(\xi_1)}\ldots\widehat{\psi(\xi_d)} \end{bmatrix}$$

In the following, we use a unique wavelet ψ to construct the tensorial wavelet Ψ, and we don't use wavelet differentiation. Then,

$$\widehat{(Id - M_\omega^\dagger A)\mathbf{u}}(\xi) = \sum_{\mathbf{j}\in\mathbb{Z}^d}\left[\sum_{\mathbf{k}\in\mathbb{Z}^d}\left(Id - M_{\omega\,\mathbf{j}}^\dagger M(\xi + 2\mathbf{k}\pi 2^\mathbf{j})\right)\widehat{\mathbf{u}}(\xi + 2\mathbf{k}\pi 2^\mathbf{j})\overline{\widehat{\Psi^*}}\left(\frac{\xi}{2^\mathbf{j}} + 2\mathbf{k}\pi 2^\mathbf{j}\right)\right]\widehat{\Psi}\left(\frac{\xi}{2^\mathbf{j}}\right) \tag{5.5}$$

This expression drives us to the following theorem:

Theorem 5.4 *Let be the set $\Theta = \{\theta = 2^\mathbf{j}\mathbf{k}; \mathbf{j}\in\mathbb{Z}^d, \mathbf{k}\in\mathbb{Z}^d\}$, with $2^\mathbf{j}\mathbf{k} = (2^{j_1}k_1, \ldots, 2^{j_d}k_d)$. Let $\|\cdot\| = \|\cdot\|_{\mathcal{M}_n(\mathbb{R})}$ be a matrix norm associated to a norm $|\cdot| = |\cdot|_{\mathbb{R}^n}$ in $\mathbb{R}^n$, and*

$$\rho(\zeta) = \sum_{\theta\in\Theta}\|\sum_{2^\mathbf{j}\mathbf{k}=\theta}\left(Id - M_{\omega\,\mathbf{j}}^\dagger M(\zeta)\right)\overline{\widehat{\Psi^*}}\left(\frac{\zeta}{2^\mathbf{j}}\right)\widehat{\Psi}\left(\frac{\zeta - 2\pi\theta}{2^\mathbf{j}}\right)\|$$

If $\exists\rho_0 < 1$ such that $\forall\zeta\in\mathbb{R}^d$, $\rho(\zeta) < \rho_0$, then

$$\|\widehat{(Id - M_\omega^\dagger A)\mathbf{u}}\|_{L^1} \le \rho_0\|\widehat{\mathbf{u}}\|_{L^1}$$

and the Richardson iteration (3.2) converges in L^∞ norm.

Proof: Given $\xi \in \mathbb{R}^d$, the series

$$\sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^d} \left(Id - M^{\dagger}_{\omega\,\mathbf{j}} M(\xi + 2\mathbf{k}\pi 2^{\mathbf{j}})\right) \widehat{\mathbf{u}}(\xi + 2\mathbf{k}\pi 2^{\mathbf{j}}) \overline{\widehat{\Psi^*}} \left(\frac{\xi}{2^{\mathbf{j}}} + 2\mathbf{k}\pi 2^{\mathbf{j}}\right) \widehat{\Psi}\left(\frac{\xi}{2^{\mathbf{j}}}\right)$$

is absolutely convergent for Ψ and Ψ^* sufficiently smooth and having enough zero moments: in this way, if $\mathbf{v} = A\mathbf{u}$, $\widehat{\mathbf{v}}(\xi) = M(\xi)\widehat{\mathbf{u}}(\xi)$, there exists $a \in \mathbb{R}$ such that

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} |\widehat{\mathbf{v}}(\xi + 2\mathbf{k}\pi 2^{\mathbf{j}})| \left|\widehat{\Psi^*}\left(\frac{\xi}{2^{\mathbf{j}}} + 2\mathbf{k}\pi 2^{\mathbf{j}}\right)\right| \le C 2^{-a(j_1 + j2 + \cdots + j_d)}$$

and $\sum_{\mathbf{j}\in\mathbb{Z}^d} 2^{-a(j_1+j2+\cdots+j_d)} \left|M^{\dagger}_{\omega\,\mathbf{j}} \widehat{\Psi}\left(\frac{\xi}{2^{\mathbf{j}}}\right)\right|$ bounded.
Then, the L^1 norm of expression (5.5) yields:

$$\begin{aligned}
&\left\|(Id - \widehat{M^{\dagger}_{\omega}A)\mathbf{u}}\right\|_{L^1} \\
&= \int_{\xi\in\mathbb{R}^d} \left|\sum_{\mathbf{j}\in\mathbb{Z}^d}\left[\sum_{\mathbf{k}\in\mathbb{Z}^d} \left(Id - M^{\dagger}_{\omega\,\mathbf{j}} M(\xi + 2\mathbf{k}\pi 2^{\mathbf{j}})\right) \widehat{\mathbf{u}}(\xi + 2\mathbf{k}\pi 2^{\mathbf{j}}) \overline{\widehat{\Psi^*}}\left(\frac{\xi}{2^{\mathbf{j}}} + 2\mathbf{k}\pi\right)\right] \widehat{\Psi}\left(\frac{\xi}{2^{\mathbf{j}}}\right)\right| d\xi \\
&= \int_{\xi\in\mathbb{R}^d} \left|\sum_{\theta\in\Theta}\left[\sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^d,\ 2^{\mathbf{j}}\mathbf{k}=\theta} \left(Id - M^{\dagger}_{\omega\,\mathbf{j}} M(\xi + 2\mathbf{k}\pi 2^{\mathbf{j}})\right) \widehat{\mathbf{u}}(\xi + 2\mathbf{k}\pi 2^{\mathbf{j}}) \overline{\widehat{\Psi^*}}\left(\frac{\xi}{2^{\mathbf{j}}} + 2\mathbf{k}\pi\right)\right] \widehat{\Psi}\left(\frac{\xi}{2^{\mathbf{j}}}\right)\right| d\xi \\
&\le \int_{\xi\in\mathbb{R}^d} \sum_{\theta\in\Theta} \left\|\sum_{2^{\mathbf{j}}\mathbf{k}=\theta} \left(Id - M^{\dagger}_{\omega\,\mathbf{j}} M(\xi + 2\mathbf{k}\pi 2^{\mathbf{j}})\right) \overline{\widehat{\Psi^*}}\left(\frac{\xi + 2\pi\theta}{2^{\mathbf{j}}}\right) \widehat{\Psi}\left(\frac{\xi}{2^{\mathbf{j}}}\right)\right\| |\widehat{\mathbf{u}}(\xi + 2\pi\theta)|\, d\xi
\end{aligned}$$

Then we make the change of variable $\zeta = \xi + 2\pi\theta$ in the integrals and we obtain:

$$\|(Id - \widehat{M^{\dagger}_{\omega}A)\mathbf{u}}\|_{L^1} \le \sum_{\theta\in\Theta}\int_{\xi\in\mathbb{R}^d} \left\|\sum_{2^{\mathbf{j}}\mathbf{k}=\theta} \left(Id - M^{\dagger}_{\omega\,\mathbf{j}} M(\zeta)\right) \overline{\widehat{\Psi^*}}\left(\frac{\zeta}{2^{\mathbf{j}}}\right) \widehat{\Psi}\left(\frac{\zeta - 2\pi\theta}{2^{\mathbf{j}}}\right)\right\| |\widehat{\mathbf{u}}(\zeta)|\, d\xi$$

This provides us with the result.
As $\|u\|_{L^\infty} \le \frac{1}{(2\pi)^d}\|\widehat{u}\|_{L^1}$, the convergence of the Richardson iteration in norm $u \mapsto \|\widehat{u}\|_{L^1}$ provides the convergence in norm L^∞. ■

Remark 5.5 *If* $M^{\dagger}_{\omega\,\mathbf{j}+(1,\ldots,1)} M(2\zeta) = M^{\dagger}_{\omega\,\mathbf{j}} M(\zeta)$*, then* $\rho(2\zeta) = \rho(\zeta)$*, and the convergence criterion can be replaced by:*
Let $D \subset \mathbb{R}^d$ *such that* $\bigcup_{j\in\mathbb{Z}} 2^j D = \mathbb{R}^d \setminus \{(0,\ldots,0)\}$*, for instance* $D = [-2\pi, 2\pi]^d \setminus [-\pi, \pi]^d$*, then* $\rho(\zeta) \le \rho_0\ \forall \zeta \in D$ *is a sufficient condition.*

6 Examples of applications – Numerical experiments

In the following section, we present different applications of our approach to the solution of operator equation. First we apply it to the Laplace equation $\Delta u = v$, and show the results of numerical experiments. Then we show how the same method improves the wavelet preconditioning of the elliptic operator $div A \nabla u = v$. Finally, we explain how to perform the Helmholtz and Craya decompositions using biorthogonal, compactly supported wavelets.

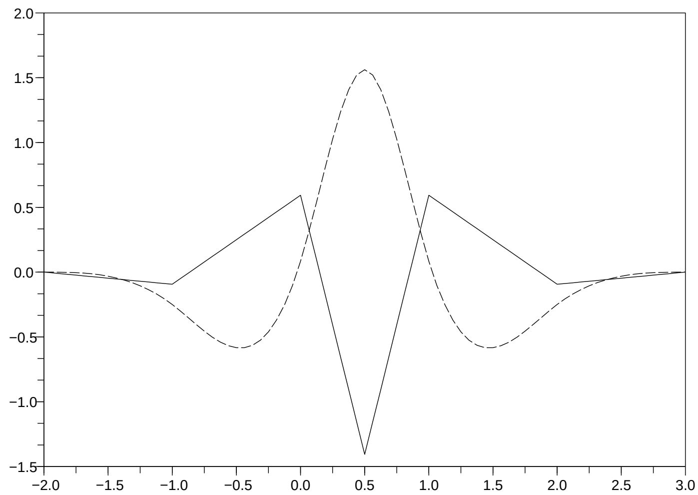

Figure 4: Spline wavelets of order two ψ_0 (plain) and order four ψ_2 (dotted) used in the numerical experiments, $\psi_2'' = 16\,\psi_0$.

6.1 The Laplace equation

We consider the Laplace equation $\Delta u = v$ on $\mathbb{R}^d$ and present and test different wavelet methods relying on the material developed in the previous sections.

The symbol of Δ is $m(\xi) = -|\xi|^2 = -\sum_{i=1}^{d} \xi_i^2$. Suppose we have constructed approximations $m^{\dagger}_{\omega\,\mathbf{j}}(\xi)$ of Δ^{-1} on the tensorial wavelet space $W_{\mathbf{j}}$ for $\mathbf{j} \in \mathbb{Z}^d$. Then, in the frame of Shannon wavelet we have to study the functions:

$$g_{\mathbf{j}}(\xi) = 1 - m(\xi) m^{\dagger}_{\omega\,\mathbf{j}}(\xi)$$

for $\mathbf{j} \in \mathbb{Z}^d$ and verify that for a given $\rho < 1$, $|g_{\mathbf{j}}(\xi)| \leq \rho$ for all $\xi \in \prod_{i=1}^{d} \pm 2^{j_i}[\pi, 2\pi]$.

We assume we have at our disposal three 1D-wavelet bases: $(\psi_{0\,jk})$, $(\psi_{1\,jk})$ and $(\psi_{2\,jk})$, and that these wavelets verify: $\psi_2' = 4\psi_1$ and $\psi_1' = 4\psi_0$.
In the theoretical computations, we assume that ψ_1 is the Shannon wavelet, while in the numerical tests, we use spline wavelets.
For more clarity, we write algorithms and make calculations in dimension two but the same results stand in any dimension. In dimension two, we have:

$$m(\xi) = -\xi_1^2 - \xi_2^2$$

We discuss and test three different wavelet algorithms:

• The first simplest way to inverse the Laplace operator is to use only one wavelet basis. This is the most classical method, advocated in [8] for instance. It corresponds to a Richardson iteration with diagonal preconditioning. It is optimal in the sense that it yields a constant rate of convergence $\rho < 1$ independent from the number of grid points. But as we will see later, other methods using wavelet differentiation give much better convergence rates. Let

$$v(x_1, x_2) = \sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^2} d_{\mathbf{jk}} \psi_{2\,j_1k_1}(x_1)\psi_{2\,j_2k_2}(x_2)$$

then to approximate the solution, we take:

$$u^{(1)}(x_1, x_2) = \sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^2} -\frac{b}{\omega_1^2 + \omega_2^2} d_{\mathbf{jk}} \psi_{2\,j_1k_1}(x_1)\psi_{2\,j_2k_2}(x_2)$$

with $\omega_i = 2^{j_i}$ and $b > 0$ a constant to fix.
Then

$$\begin{aligned} v^{(1)}(x_1, x_2) &= v(x_1, x_2) - \Delta u^{(1)}(x_1, x_2) \\ &= v(x_1, x_2) - \sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^2} -b\frac{16\omega_1^2}{\omega_1^2 + \omega_2^2} d_{\mathbf{jk}} \psi_{0\,j_1k_1}(x_1)\psi_{2\,j_2k_2}(x_2) - b\frac{16\omega_2^2}{\omega_1^2 + \omega_2^2} d_{\mathbf{jk}} \psi_{2\,j_1k_1}(x_1)\psi_{0\,j_2k_2}(x_2) \end{aligned}$$

We iterate the process (3.7).
The symbol corresponding to this wavelet approximation of Δ^{-1} is:

$$m^{\dagger}_{\omega_{\mathbf{j}}}(\xi) = -\frac{b}{\omega_1^2 + \omega_2^2}$$

hence

$$g_{\mathbf{j}}(\xi) = 1 - b\frac{\xi_1^2 + \xi_2^2}{\omega_1^2 + \omega_2^2}$$

for $\xi_i \in \pm[\pi\omega_i, 2\pi\omega_i]$ with Shannon wavelet, that means that

$$1 - 4\pi^2 b \leq g_{\mathbf{j}}(\xi) \leq 1 - \pi^2 b$$

then the optimal value is $b = \frac{2}{5\pi^2} \sim 0.0405$. For this value of b, $\rho = \frac{3}{5} = 0.6$. This rate is the best convergence rate we can obtain with any wavelet basis. In practice, the convergence rate observed with order 4 spline wavelets (Figure 4) is $\rho = 0.84$ for $b = 0.0187$.

• The second one is very similar to the first one but utilizes the differentiation of wavelet. Let

$$v(x_1, x_2) = \sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^2} d_{\mathbf{jk}} \psi_{0\,j_1k_1}(x_1)\psi_{0\,j_2k_2}(x_2)$$

then we take

$$u^{(1)}(x_1, x_2) = \sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^2} -\frac{b}{\omega_1^2 + \omega_2^2} d_{\mathbf{jk}} \psi_{2\,j_1k_1}(x_1)\psi_{2\,j_2k_2}(x_2)$$

for some $b > 0$. Knowing that $-\xi^2\widehat{\psi}_{2\,jk} = 16\omega^2\widehat{\psi}_{0\,jk}$, the associated symbol is given by:

$$m^{\dagger}_{\omega\,\mathbf{j}}(\xi) = -\frac{16^2\omega_1^2\omega_2^2 b}{(\omega_1^2 + \omega_2^2)\xi_1^2\xi_2^2}$$

then

$$g_{\mathbf{j}}(\xi) = 1 - b\frac{16^2\omega_1^2\omega_2^2}{\omega_1^2 + \omega_2^2}\left(\frac{1}{\xi_1^2} + \frac{1}{\xi_2^2}\right)$$

and as $\xi_i^2 \in [\pi^2\omega_i^2, 4\pi^2\omega_i^2]$, we obtain, for wavelets of Shannon type:

$$1 - b\frac{16^2}{\pi^2} \leq g_{\mathbf{j}}(\xi) \leq 1 - b\frac{16^2}{4\pi^2}$$

the optimal convergence rate is $\rho = \frac{3}{5}$ for $b = \frac{2}{5}\frac{4\pi^2}{16^2} \sim 0.0617$. With ψ_2 spline wavelet of order 4 and ψ_0 spline wavelet of order 2 (Figure 4), we observe the convergence rate $\rho = 0.75$ for $b = 0.0594$.

• Finally, a more equilibrated method consists in making two different wavelet decompositions of v:

$$v(x_1, x_2) = \sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^2} d_{\mathbf{jk}}^{(1)} \psi_{0\,j_1k_1}(x_1)\psi_{2\,j_2k_2}(x_2) = \sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^2} d_{\mathbf{jk}}^{(2)} \psi_{2\,j_1k_1}(x_1)\psi_{0\,j_2k_2}(x_2)$$

then we take as a first iterate

$$u^{(1)}(x_1, x_2) = \sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^2} -b\frac{\omega_1^2 d_{\mathbf{jk}}^{(1)} + \omega_2^2 d_{\mathbf{jk}}^{(2)}}{(\omega_1^2 + \omega_2^2)^2} \psi_{2\,j_1k_1}(x_1)\psi_{2\,j_2k_2}(x_2)$$

This corresponds to the symbol:

$$m_{\omega\,\mathbf{j}}^{\dagger}(\xi) = -\frac{16\,b}{(\omega_1^2 + \omega_2^2)^2}\left(\frac{\omega_1^4}{\xi_1^2} + \frac{\omega_2^4}{\xi_2^2}\right)$$

then for $\xi_i = \omega_i\zeta_i$, $\zeta_i \in \pm[\pi, 2\pi]$ for wavelets of type Shannon, we have:

$$g_{\mathbf{j}}(\xi) = 1 - 16\,b\left(\frac{\omega_1^2}{\omega_1^2 + \omega_2^2}\zeta_1^2 + \frac{\omega_2^2}{\omega_1^2 + \omega_2^2}\zeta_2^2\right)\left(\frac{\omega_1^2}{\omega_1^2 + \omega_2^2}\frac{1}{\zeta_1^2} + \frac{\omega_2^2}{\omega_1^2 + \omega_2^2}\frac{1}{\zeta_2^2}\right)$$

The Kantorovitch inequality states that

$$\max_{\alpha_i, \sum\alpha_i^2 = 1}\left(\sum_{i=1}^{d}\alpha_i^2\zeta_i^2\right)\left(\sum_{i=1}^{d}\frac{\alpha_i^2}{\zeta_i^2}\right) = \frac{1}{4}\left(\frac{\min_i|\zeta_i|}{\max_i|\zeta_i|} + \frac{\max_i|\zeta_i|}{\min_i|\zeta_i|}\right)^2$$

and we have from Schwartz inequality $\|\alpha\|_2\|\beta\|_2 \geq \langle\alpha, \beta\rangle$,

$$\min_{\alpha_i, \sum\alpha_i^2 = 1}\left(\sum_{i=1}^{d}\alpha_i^2\zeta_i^2\right)\left(\sum_{i=1}^{d}\frac{\alpha_i^2}{\zeta_i^2}\right) = 1$$

Then as $|\zeta_i| \in [\pi, 2\pi]$, $\frac{1}{4}\left(\frac{\min_i|\zeta_i|}{\max_i|\zeta_i|} + \frac{\max_i|\zeta_i|}{\min_i|\zeta_i|}\right)^2 = \frac{1}{4}\left(\frac{1}{2} + 2\right)^2 = \frac{25}{16}$, and

$$1 - 25\,b \leq g_{\mathbf{j}}(\xi) \leq 1 - 16\,b$$

the optimal choice for b is: $b = \frac{2}{41} \sim 0.0488$. For this value of b, the convergence rate –for Shannon wavelets– is $\rho = \frac{9}{41} \sim 0.22$ which is three times better than $\frac{3}{5}$: $\left(\frac{3}{5}\right)^3 \sim \frac{9}{41}$.
In practice, with spline wavelets of order 4 and 2 (Figure 4) we obtain $\rho = 0.48$ for $b = 0.045$.

Remark 6.1 *This latest approximation shall be linked with vaguelette decomposition since we iteratively expand* v *in the vaguelette basis* $\Delta\psi_2\psi_2$*:*

$$v(x_1, x_2) = \sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^2} d_{\mathbf{jk}} \left(2^{2j_1}\psi_{0\,j_1k_1}(x_1)\psi_{2\,j_2k_2}(x_2) + 2^{2j_2}\psi_{2\,j_1k_1}(x_1)\psi_{0\,j_2k_2}(x_2)\right)$$

in order to integrate the Laplace operator $\Delta u = v$ *with*

$$u(x_1, x_2) = \frac{1}{16}\sum_{\mathbf{j},\mathbf{k}\in\mathbb{Z}^2} d_{\mathbf{jk}}\psi_{2\,j_1k_1}(x_1)\psi_{2\,j_2k_2}(x_2)$$

The computational cost of the complete solution with wavelet multigrid methods using Richardson iteration are:

(number of points) × (number of wavelet transform per iteration)
× (size of the wavelet filters) × (number of iterations)

Hence generally speaking, its cost is $O(N\log(\varepsilon))$ where N is the number of points and ε the admissible error.

For instance, if we use the spline wavelet of fig 4, the size of the wavelet filters is 10, the number of iterations to reach an accuracy of 10^{-6} is $\log(10^{-6})/\log(\rho)$. Then as we need three wavelet transforms (direct or inverse) in two directions, with respectively $\rho = 0.84$ and $\rho = 0.75$ for the first and second methods, and four wavelet transforms in two directions with $\rho = 0.48$ for the third method, it means that the total cost of each method is:

- $\sim 4800N$ for the first method,
- $\sim 2900N$ for the second method,
- $\sim 1500N$ for the third method.

In addition these costs, we have to take into account the cost of reprojection in different MRA's. nevertheless we claim that this is negligible.

The three methods were tested in the case:

$$v(x_1, x_2) = \cos(6\pi x_1)\sin(10\pi x_2)$$

on $[0,1]^2$ with periodic boundary conditions and a 256^2 grid (at scale $j = 8$). The results are displayed on figure 5. We plotted the L^2-norm of the residual $(v - \Delta u_n)$ after n iterations in the algorithm (3.2). We notice that the third method using wavelet differentiation outperforms the two others, the second one using two different wavelet bases, being more efficient than the classical method which uses only one type of wavelet.

6.2 A more general elliptic case

Now we consider the more general elliptic equation on $H^2(\mathbb{R}^d)$:

$$\mathrm{div} A\nabla u = v \tag{6.1}$$

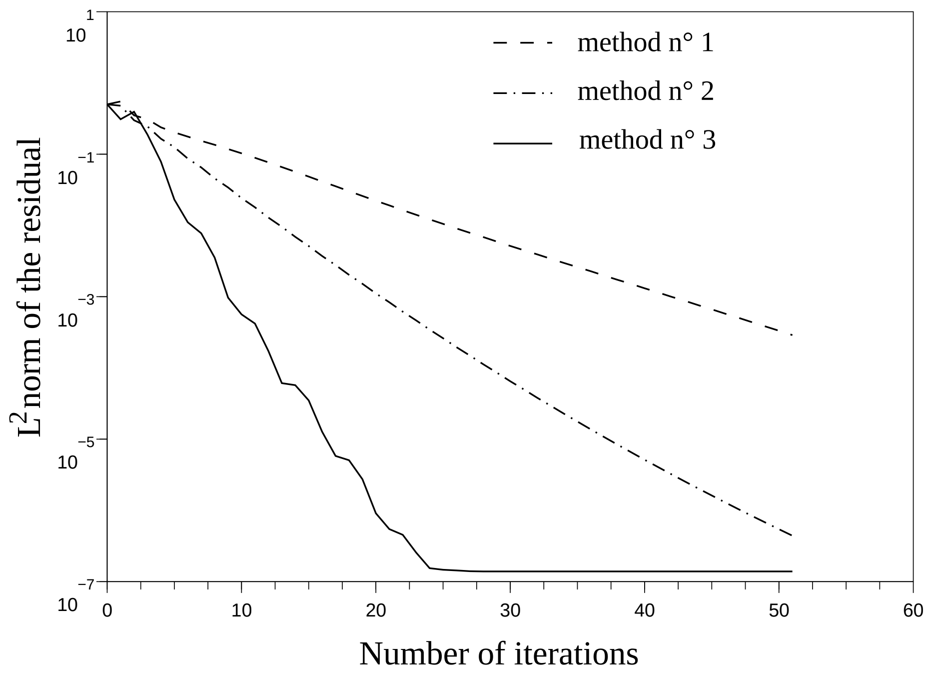

Figure 5: Numerical experiment of the three methods: $\psi_2\psi_2 \to \psi_2\psi_2$, $\psi_0\psi_0 \to \psi_2\psi_2$ and $(\psi_2\psi_0, \psi_0\psi_2) \to \psi_2\psi_2$ with spline wavelets (fig 4).

Without any loss of generality, we assume that $A = (a_{ij})$ is symmetric, then the symbol corresponding to the operator (6.1) is

$$m(\xi) = -\sum_{i=1}^{d} a_{ii}\xi_i^2 - \sum_{1\le i<j\le d} a_{ij}\xi_i\xi_j$$

We propose to study two particular cases: $a_{ij} = 0$ for $i \neq j$ in dimension d, and the case of the dimension 2. A generalization to larger dimensions is still under consideration. Moreover, to simplify the computations, we only consider Shannon wavelets.

First, in the case when A is diagonal we will study the three methods of the previous subsection 6.1.

- The first 'classical' method corresponds to the symbol:

$$m^{\dagger}_{\omega\mathbf{j}}(\xi) = -\frac{b}{\sum_{i=1}^{d} a_{ii}\pi^2\omega_i^2} \qquad \text{for} \quad \xi_i \in \pm[\pi\omega_i, 2\pi\omega_i],\ i \in [1, d]$$

It uses only one wavelet basis. The number b is a non negative constant, chosen equal to $\frac{2}{5}$ the optimal value for Shannon wavelet, as in the previous section . Then

$$g_{\mathbf{j}}(\xi) = 1 - m(\xi)m^{\dagger}_{\omega\mathbf{j}}(\xi) \in \left[-\frac{3}{5}, \frac{3}{5}\right]$$

which leads to a convergence rate of $\frac{3}{5}$ as in dimension two.

• The generalization of the second method to larger dimensions does not come out with a good convergence rate. Its symbol is given by:

$$m^{\dagger}_{\omega\mathbf{j}}(\xi) = -\frac{b\ \prod_{i=1}^{d}\pi^2\omega_i^2}{\sum_{i=1}^{d}\pi^2 a_{ii}\omega_i^2 \prod_{i=1}^{d}\xi_i^2} \qquad \text{for} \quad \xi_i \in \pm[\pi\omega_i, 2\pi\omega_i],\ i \in [1,d]$$

Then for $b = \frac{2}{4^{d-1}+1}$, the optimal convergence rate is $\frac{4^{d-1}-1}{4^{d-1}+1}$, which behaves badly for large d. For instance, as soon as $d = 3$, the convergence rate falls to $\frac{15}{17}$.

• On the contrary, the third method still exhibits a very good convergence rate. The generalization to dimension d of the third method gives the following symbol:

$$m^{\dagger}_{\omega\mathbf{j}}(\xi) = -\frac{b}{\left(\sum_{i=1}^{d} a_{ii}\omega_i^2\right)^2}\left(\sum_{i=1}^{d} a_{ii}\frac{\omega_i^4}{\xi_i^2}\right) \qquad \text{for} \quad \xi_i \in \pm[\pi\omega_i, 2\pi\omega_i],\ i \in [1,d]$$

Then, if we take, as in subsection 6.1, $\xi_i = \omega_i\zeta_i$, with $\zeta_i \in \pm[\pi, 2\pi]$, and α_i

$$\alpha_i^2 = \frac{a_{ii}\omega_i^2}{\sum_{j=1}^{d} a_{jj}\omega_j^2}$$

we obtain

$$g_{\mathbf{j}}(\xi) = 1 - m(\xi)m^{\dagger}_{\omega\mathbf{j}}(\xi) = 1 - b\left(\sum_{i=1}^{d}\alpha_i^2\zeta_i^2\right)\left(\sum_{i=1}^{d}\alpha_i^2\zeta_i^{-2}\right)$$

and we apply the Kantorovitch inequality. As in dimension two, this gives a convergence rate equal to $\frac{9}{41}$ for $b = \frac{32}{41}$.

The two-dimensional case

Now, we are interested in the elliptic operator ($\text{div}A\nabla$) in dimension two. Its symbol is given by

$$m(\xi) = -(a_{11}\xi_1^2 + a_{22}\xi_2^2 + 2a_{12}\xi_1\xi_2) \tag{6.2}$$

with $a_{11}a_{22} > a_{12}^2$. We introduce the quantity:

$$\gamma = \frac{a_{12}}{\sqrt{a_{11}a_{22}}}$$

that we assume positive: $a_{12} \geq 0$, without any loss of generality. We apply to this equation, the first –classical– method and the third using the differentiation of wavelet bases.

• For the usual diagonal preconditioning $m^{\dagger}_{\omega\mathbf{j}}(\xi) = -b(\omega)$ with $b(\omega) > 0$, the optimal convergence rate we can obtain depends on $\max(-m(\xi))$ and $\min(-m(\xi))$ for $\xi_i \in \pm[\pi\omega_i, 2\pi\omega_i],\ i \in [1,d]$, since:

$$g_{\mathbf{j}}(\xi) = 1 - m(\xi)m^{\dagger}_{\omega\mathbf{j}}(\xi) = 1 + b(\omega)m(\xi) \in [1 - b(\omega)\max(-m(\xi)), 1 - b(\omega)\min(-m(\xi))]$$

The optimal $b(\omega)$ is

$$b(\omega) = \frac{2}{\max(-m(\xi)) + \min(-m(\xi))}$$

For this value of $b(\omega)$, we have the optimal convergence rate

$$\rho = \frac{\max(-m(\xi) - \min(-m(\xi))}{\max(-m(\xi)) + \min(-m(\xi))}$$

Then as for (6.2), we have

$$\max(-m(\xi)) = -m(2\omega_1, 2\omega_2) = 4(a_{11}\omega_1^2 + a_{22}\omega_2^2 + 2a_{12}\omega_1\omega_2)$$

and

$$\min(-m(\xi)) \leq -m(\omega_1, -\omega_2) = a_{11}\omega_1^2 + a_{22}\omega_2^2 - 2a_{12}\omega_1\omega_2$$

then for $\alpha_i = \omega_i\sqrt{a_{ii}}$

$$\rho \geq \frac{3\alpha_1^2 + 3\alpha_2^2 + 10\gamma\alpha_1\alpha_2}{5\alpha_1^2 + 5\alpha_2^2 + 6\gamma\alpha_1\alpha_2}$$

As $Q(x) = \frac{3x^2+10\gamma x+3}{5x^2+3\gamma x+5}$ has its maximum at $x = 1$, then the convergence rate of the classical method is, at best:

$$\rho = \frac{3 + 5\gamma}{5 + 3\gamma}$$

- As an alternative to the previous method, we propose a wavelet preconditioning based on the third method of subsection 6.1:

$$m^\dagger_{\omega\mathbf{j}}(\xi) = -b(\omega)\left(a_{11}\frac{\omega_1^4}{\xi_1^2} + a_{22}\frac{\omega_2^4}{\xi_2^2} - 2a_{12}\frac{\omega_1^2\omega_2^2}{\xi_1\xi_2}\right)$$

Then the convergence rate depends on $\max(m(\xi)m^\dagger_{\omega\mathbf{j}}(\xi))$ and $\min(m(\xi)m^\dagger_{\omega\mathbf{j}}(\xi))$, for $\xi_i \in \pm[\omega_i, 2\omega_i],\ i \in [1, d]$. Let $\xi_i = \omega_i\zeta_i$, $\zeta_i \in \pm[\pi, 2\pi]$, we write

$$\begin{aligned} b(\xi) &= 1 - b(\omega)(a_{11}\xi_1^2 + a_{22}\xi_2^2 + 2a_{12}\xi_1\xi_2)\left(a_{11}\frac{\omega_1^4}{\xi_1^2} + a_{22}\frac{\omega_2^4}{\xi_2^2} - 2a_{12}\frac{\omega_1^2\omega_2^2}{\xi_1\xi_2}\right) \\ &= 1 - b(\omega)\left[(a_{11}\omega_1^2\zeta_1^2 + a_{22}\omega_2^2\zeta_2^2 + 2a_{12}\omega_1\omega_2\zeta_1\zeta_2)\left(a_{11}\frac{\omega_1^2}{\zeta_1^2} + a_{22}\frac{\omega_2^2}{\zeta_2^2} - 2a_{12}\frac{\omega_1\omega_2}{\zeta_1\zeta_2}\right)\right] \\ &= 1 - b(\omega)\left[(a_{11}^2\omega_1^4 + a_{22}^2\omega_2^4 + a_{11}a_{22}\omega_1^2\omega_2^2\left(\frac{\zeta_1^2}{\zeta_2^2} + \frac{\zeta_2^2}{\zeta_1^2}\right) - 4a_{12}^2\omega_1^2\omega_2^2 \right. \\ &\quad \left. +2a_{12}\omega_1\omega_2\left(a_{11}\omega_1^2 - a_{22}\omega_2^2\right)\left(\frac{\zeta_2}{\zeta_1} - \frac{\zeta_1}{\zeta_2}\right)\right] \end{aligned}$$

Let $X = \left(\frac{\zeta_2}{\zeta_1} - \frac{\zeta_1}{\zeta_2}\right) \in [-\frac{3}{2}, \frac{3}{2}]$, then $X^2 = \left(\frac{\zeta_1^2}{\zeta_2^2} + \frac{\zeta_2^2}{\zeta_1^2}\right) - 2$. As a result, the expression inside the square brackets reads:

$$P(X) = a_{11}a_{22}\omega_1^2\omega_2^2X^2 + 2a_{12}\omega_1\omega_2\left(a_{11}\omega_1^2 - a_{22}\omega_2^2\right)X + \left(a_{11}\omega_1^2 + a_{22}\omega_2^2\right)^2 - 4a_{12}^2\omega_1^2\omega_2^2$$

As previously, we introduce $\alpha_i = \omega_i\sqrt{a_{ii}}$, then

$$\omega_1\omega_2a_{12} = \omega_1\omega_2\gamma\sqrt{a_{11}a_{22}} = \gamma\alpha_1\alpha_2$$

Then

$$P(X) = \alpha_1^2\alpha_2^2X^2 + 2\gamma\alpha_1\alpha_2(\alpha_1^2 - \alpha_2^2)X + (\alpha_1^2 + \alpha_2^2)^2 - 4\gamma^2\alpha_1^2\alpha_2^2$$

The minimum of $P(X)$ on $\mathbb{R}$ is attained for $X_m = \frac{\gamma(\alpha_2^2-\alpha_1^2)}{\alpha_1\alpha_2}$, and

$$P(X_m) = (\alpha_1^2+\alpha_2^2)^2(1-\gamma^2)$$

and its maximum on $[-\frac{3}{2}, \frac{3}{2}]$ at $-\frac{3}{2}$ or $\frac{3}{2}$. Without loss of generality, we assume $\alpha_1 \geq \alpha_2$, then $P(X)$ reaches its maximum at $X_M = \frac{3}{2}$, and

$$\begin{aligned} P(X_M) &= \frac{9}{4}\alpha_1^2\alpha_2^2 + 3\gamma\alpha_1\alpha_2(\alpha_1^2-\alpha_2^2) + (\alpha_1^2+\alpha_2^2)^2 - 4\gamma^2\alpha_1^2\alpha_2^2 && (6.3)\\ &= \left(\frac{9}{4} - 4\gamma^2\right)\alpha_1^2\alpha_2^2 + 3\gamma\alpha_1\alpha_2(\alpha_1^2-\alpha_2^2) + (\alpha_1^2+\alpha_2^2)^2 && (6.4)\end{aligned}$$

Let $e = \frac{\alpha_1^2-\alpha_2^2}{\alpha_1^2+\alpha_2^2} \in [0,1)$, then $\alpha_1^2\alpha_2^2 = \frac{1}{4}(\alpha_1^2+\alpha_2^2)^2(1-e^2)$, and

$$P(X_M) = (\alpha_1^2+\alpha_2^2)^2\left(\left(\frac{9}{16}-\gamma^2\right)(1-e^2) + \frac{3}{2}\gamma e\sqrt{1-e^2}+1\right)$$

For $e = \cos\theta$, $\theta \in [0, \frac{\pi}{2}]$, $(1-e^2) = \frac{1-\cos(2\theta)}{2}$ and $e\sqrt{1-e^2} = \frac{\sin(2\theta)}{2}$ then

$$P(X_M) = (\alpha_1^2+\alpha_2^2)^2\left(\frac{1}{2}\left(\frac{9}{16}-\gamma^2\right) + 1 + \frac{1}{2}\left(\frac{9}{16}-\gamma^2\right)\cos(2\theta) + \frac{3}{4}\gamma\sin(2\theta)\right)$$

As $\max_\theta(a\cos\theta + b\sin\theta) = (a^2+b^2)^{1/2}$, after simplification, we obtain:

$$P(X_M) \leq \frac{25}{16}(\alpha_1^2+\alpha_2^2)^2$$

Then, for a well chosen $b(\omega)$,

$$|g_{\mathbf{j}}(\xi)| = |1 - m(\xi)m^{\dagger}_{\omega\,\mathbf{j}}(\xi)| \leq \frac{P(X_M)-P(X_m)}{P(X_M)+P(X_m)}$$

Hence we obtain

$$\rho = \frac{9+16\gamma^2}{41-16\gamma^2}$$

The curves $\frac{3+5\gamma}{5+3\gamma}$ and $\frac{9+16\gamma^2}{41-16\gamma^2}$ corresponding respectively to the classical method and to the new one are plotted in figure 6. One can notice the better performances of the latest method compared to the first one.

Remark 6.2 *Hence, the spectral analysis of wavelet algorithm allows not only to demonstrate the convergence, but it also permits to construct new algorithms adapted to specific differential problems.*

6.3 The Helmholtz and Craya decompositions

The Helmholtz decomposition consists in separating the divergence-free part of a vector-valued function from its gradient part. In spectral space, it reads:

$$\hat{\mathbf{u}} = \widehat{\mathbb{P}\mathbf{u}} + \widehat{\mathbb{Q}\mathbf{u}}$$

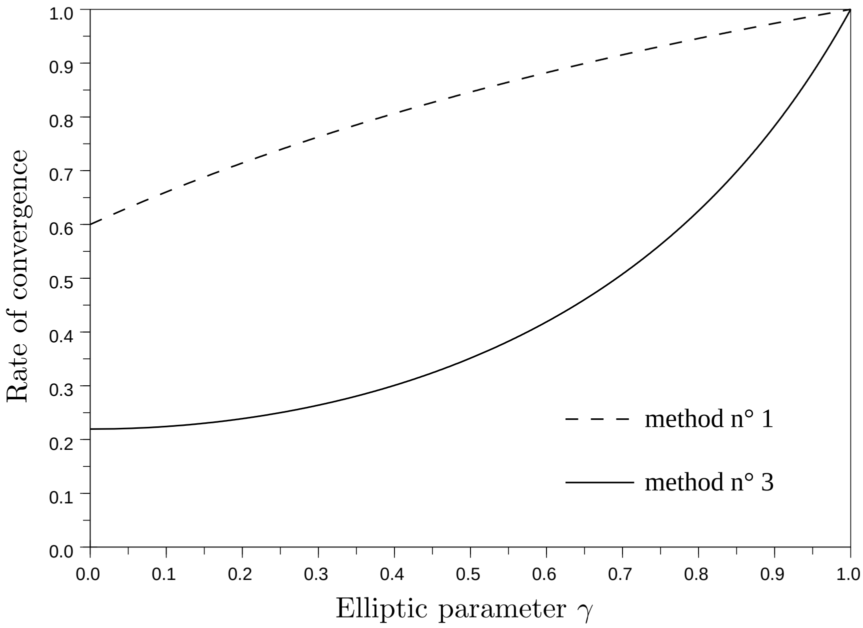

Figure 6: The convergence rates obtained for the classical method (n°1) and the method using wavelet differentiation (n°3) for solving the elliptic equation (6.1) in dimension two with non-diagonal A.

with, in dimension d,

$$\widehat{\mathbb{P}\mathbf{u}} = \left(Id - \frac{1}{\sum_{\ell=1}^{d} \xi_\ell^2} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_d \end{bmatrix} \times \begin{bmatrix} \xi_1 & \dots & \xi_d \end{bmatrix} \right) \hat{\mathbf{u}}$$

and

$$\widehat{\mathbb{Q}\mathbf{u}} = \frac{1}{\sum_{\ell=1}^{d} \xi_\ell^2} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_d \end{bmatrix} \times \begin{bmatrix} \xi_1 & \dots & \xi_d \end{bmatrix} \hat{\mathbf{u}}$$

To make the Helmholtz decomposition of vector fields with wavelets, [15] proposes to project the function to decompose on wavelet divergence-free and gradient spaces. As this projection is not orthogonal, we iterate it in a process similar to the one presented in this paper.

If we note $\omega_\ell = 2^{\mathbf{j}_\ell}$ and $\zeta_\ell = \frac{\omega_\ell^2}{\xi_\ell}$, this Helmholtz wavelet decomposition is written with

$$\widehat{P_{\text{div}}\mathbf{u}} = \left(Id - \frac{1}{\sum_{\ell=1}^{d} \omega_\ell^2} \begin{bmatrix} \zeta_1 \\ \vdots \\ \zeta_d \end{bmatrix} \times \begin{bmatrix} \xi_1 & \dots & \xi_d \end{bmatrix} \right) \hat{\mathbf{u}}$$

and

$$\widehat{Q_{\text{rot}}\mathbf{u}} = \frac{1}{\sum_{\ell=1}^{d} \omega_\ell^2} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_d \end{bmatrix} \times \begin{bmatrix} \zeta_1 & \dots & \zeta_d \end{bmatrix} \hat{\mathbf{u}}$$

These projectors are written with the algebra generated by wavelet transforms and wavelet differentiation of theorem 4.1. The divergence-free constraint is embedded in the choice of the wavelet bases and is verified exactly. In [15], we prove that the iterative sequential projection of the residual with $(Id - Q_{\text{rot}})(Id - P_{\text{div}})$ provides a Helmholtz wavelet decomposition. The same result stands for the simultaneous projection $(Id - P_{\text{div}} - Q_{\text{rot}})$.

In dimension 3, the projector on divergence-free functions can be put in the form:

$$\widehat{P_{\text{div}}\mathbf{u}} = \frac{1}{|\omega|^2}\begin{bmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & \zeta_3 & -\zeta_2 \\ -\zeta_3 & 0 & \zeta_1 \\ \zeta_2 & -\zeta_1 & 0 \end{bmatrix} \hat{\mathbf{u}}$$

with $|\omega|^2 = \sum_{\ell=1}^{3} \omega_\ell^2$. It corresponds to the projection onto the vector wavelets

$$\Psi_{\text{div},1} = \begin{bmatrix} 0 \\ \omega_3 \psi_{0\mathbf{jk}}\psi_{1\mathbf{jk}}\psi_{0\mathbf{jk}} \\ -\omega_2 \psi_{0\mathbf{jk}}\psi_{0\mathbf{jk}}\psi_{1\mathbf{jk}} \end{bmatrix},\ \Psi_{\text{div},2} = \begin{bmatrix} -\omega_3 \psi_{1\mathbf{jk}}\psi_{0\mathbf{jk}}\psi_{0\mathbf{jk}} \\ 0 \\ \omega_1 \psi_{0\mathbf{jk}}\psi_{0\mathbf{jk}}\psi_{1\mathbf{jk}} \end{bmatrix} \text{ and } \Psi_{\text{div},3} = \begin{bmatrix} \omega_2 \psi_{1\mathbf{jk}}\psi_{0\mathbf{jk}}\psi_{0\mathbf{jk}} \\ -\omega_1 \psi_{0\mathbf{jk}}\psi_{1\mathbf{jk}}\psi_{0\mathbf{jk}} \\ 0 \end{bmatrix}$$

as these wavelets form a redundant basis we can discriminate the horizontal components and make a linear combination of the two wavelets having a vertical component:

$$\Psi_{\sim T} = 2^{j_1}\Psi_{\text{div},2} + 2^{j_2}\Psi_{\text{div},1}$$

With the notations $|\omega|^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$ and $|\omega^*|^2 = \omega_1^2 + \omega_2^2$, it corresponds to writing the wavelet projector P_{div} in the equivalent form:

$$\widehat{P_{\text{div}}\mathbf{u}} = \begin{bmatrix} \xi_2 & \omega_3\zeta_1 \\ -\xi_1 & \omega_3\zeta_2 \\ 0 & -|\omega^*|^2\frac{\zeta_3}{\omega_3} \end{bmatrix} \times \begin{bmatrix} \frac{1}{|\omega^*|^2} & 0 \\ 0 & \frac{1}{|\omega^*|^2|\omega|^2} \end{bmatrix} \times \begin{bmatrix} \zeta_2 & -\zeta_1 & 0 \\ \omega_3\xi_1 & \omega_3\xi_2 & -|\omega^*|^2\frac{\xi_3}{\omega_3} \end{bmatrix} \hat{\mathbf{u}} \quad (6.5)$$

But this decomposition is not orthogonal. The exact orthogonal decomposition between horizontal divergence-free functions and its divergence-free orthogonal complement –which has to be gradient in the horizontal plans– is called the Craya decomposition. The Craya decomposition is a partition of the divergence-free 3-dimensional vector functions into its *toroidal* part, and its *poloidal* part. With the help of Fourier transform, it consist in

$$\hat{\mathbf{u}} = \widehat{\mathbb{P}_{\mathrm{T}}\mathbf{u}} + \widehat{\mathbb{P}_{\mathrm{P}}\mathbf{u}} + \widehat{\mathbb{Q}\mathbf{u}}$$

where the orthogonal projector $\mathbb{Q}$ is the same as for the Helmholtz decomposition and we write the toroidal projector $\mathbb{P}_{\mathrm{T}}$ and poloidal projector $\mathbb{P}_{\mathrm{P}}$:

$$\widehat{\mathbb{P}_{\mathrm{T}}\mathbf{u}} = \frac{1}{|\xi^*|^2}\begin{bmatrix} \xi_2 \\ -\xi_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} \xi_2 & -\xi_1 & 0 \end{bmatrix} \hat{\mathbf{u}},\ \widehat{\mathbb{P}_{\mathrm{P}}\mathbf{u}} = \frac{1}{|\xi^*|^2|\xi|^2}\begin{bmatrix} \xi_1\xi_3 \\ \xi_2\xi_3 \\ -|\xi^*|^2 \end{bmatrix} \times \begin{bmatrix} \xi_1\xi_3 & \xi_2\xi_3 & -|\xi^*|^2 \end{bmatrix} \hat{\mathbf{u}}$$

with the notations $|\xi|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$ and $|\xi^*|^2 = \xi_1^2 + \xi_2^2$. We propose to project the function $\mathbf{u}$ on the toroidal space using $\Psi_{\text{div,T}} = \Psi_{\text{div},3}$ and a biorthogonal projection $P_{\text{div,T}}$ proceeding as in (6.5), first column of the first matrix and first column of the second matrix. We operate the poloidal projection $P_{\text{div,P}}$ thanks to the vector vaguelette:

$$\Psi_{\text{div,P}} = \begin{bmatrix} \omega_1\omega_3\psi_{1\mathbf{jk}}\psi_{2\mathbf{jk}}\psi_{0\mathbf{jk}} \\ \omega_2\omega_3\psi_{2\mathbf{jk}}\psi_{1\mathbf{jk}}\psi_{0\mathbf{jk}} \\ -(\omega_1^2\psi_{0\mathbf{jk}}\psi_{2\mathbf{jk}}\psi_{1\mathbf{jk}} + \omega_2^2\psi_{2\mathbf{jk}}\psi_{0\mathbf{jk}}\psi_{1\mathbf{jk}}) \end{bmatrix},$$

and writing:

$$\widehat{\mathbb{P}_\mathrm{P}\mathbf{u}} = \frac{1}{|\omega^*|^2|\omega|^2}\left[\begin{array}{c}\xi_1\xi_3\\ \xi_2\xi_3\\ -|\xi^*|^2\end{array}\right]\times\left[\begin{array}{ccc}\zeta_1\zeta_3 & \zeta_2\zeta_3 & -|\zeta^*|^2\end{array}\right]\widehat{\mathbf{u}}$$

with the notation $|\zeta^*|^2 = \zeta_1^2+\zeta_2^2$. The results provided by this wavelet Craya decomposition will be presented in a further work.

Remark 6.3 *As shown in [15], tensorial wavelets are not compulsory for such methods. It readily extends to MRAs or partially anisotropic wavelets –intermediate between tensorial and MRA.*

7 Conclusion

In this paper, we presented a multi-grid method based on wavelet decomposition, and we estimated the convergence rate of the associated Richardson iteration. It is a simplified –in the sense that we used FWT and not vaguelettes– or improved version of numerical methods already widely in use for solving PDE's [19, 21, 25, 7, 8, 9]. It relies on the differentiation of wavelet [22, 26] to extend computational methods using FWT.

This work also provides an original point of view on these wavelet algorithms by using the notion of symbol of an operator from the operator theory, and Shannon wavelets.

The main results of this paper are: the computation of general conditions for the convergence of wavelet algorithms, including precise conditions for Shannon wavelets; the theoretical construction of wavelet algorithms in order to approximate operators with constant coefficients; the exact computation of the convergence rate for Laplace equation and its improvement by wavelet differentiation. It investigates the use of wavelet packets. Indeed, wavelet packets seem to provide a powerful solver for many kinds of PDE's but their use is still theoretical and prospective.

The perspective is to extend these results to wavelets on the interval and to construct efficient wavelet packets which can be used numerically. An interesting perspective will be to reproduce these computations in the case of operators with non constant coefficients

These algorithms have been used successfully to simulate the Navier-Stokes equations (see [12] and [16]).

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