Wavelet methods

For the numerical simulation of incompressible fluids

Erwan Deriaz

Erwan.Deriaz@imag.fr

Laboratoire de Modélisation et Calcul (Grenoble)

PhD Director : **Valérie Perrier**

Universität Ulm, Abteilung Numerik, Seminar

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- **Wavelets for the Navier-Stokes equations**

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad , \quad \mathrm{div}(\mathbf{u}) = 0$$

Decomposition of $\mathbf{u}$ in a Riesz basis of $\mathbf{H}_{div,0}$ [Urban96]:

$$\mathbf{u} = \sum_{k\in\mathbb{Z}^d} c_k(t)\phi_k^{df} + \sum_{j\geq 0}\sum_{k\in\mathbb{Z}^d} d_{j,k}(t)\psi_{j,k}^{df}$$

The Navier Stokes equations can be projected on $\mathbf{H}_{div,0}$:

$$\partial_t \mathbf{u} + \mathbb{P}((\mathbf{u}\cdot\nabla)\mathbf{u}) - \nu\Delta\mathbf{u} = \mathbb{P}(\mathbf{f})$$

where we noted $\mathbb{P}$ the Leray projector.

• Perspectives: **Numerical resolution** of incompressible Navier-Stokes equations in dimension 2 and 3 by an adaptive method:

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla \mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + f \qquad t \geq 0,\ x \in]0,1[^n\ n=2 \text{ or } 3 \\ \text{div } \mathbf{u} = \nabla \cdot \mathbf{u} = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \\ \mathbf{u}(x,0) = \mathbf{u}^0(x) \\ +\text{limit conditions (periodic, Dirichlet homogeneous or non homogeneous)} \end{cases}$$

With a wavelet discretization:

$$\mathbf{u}(x,t^n) \approx \mathbf{u}_N(x,t^n) = \sum_{\alpha \in A_n} c_\alpha^n \, \Psi_\alpha(x)$$

with $Card(A_n) = N$ and $\Psi_\alpha \in \mathrm{H}_{\mathrm{div},0} = \{\mathbf{u} \in L^2 / \mathrm{div}(\mathbf{u}) = 0\}$.

- **Plan** :

I - Divergence-free wavelets

II - Helmholtz decomposition with wavelets

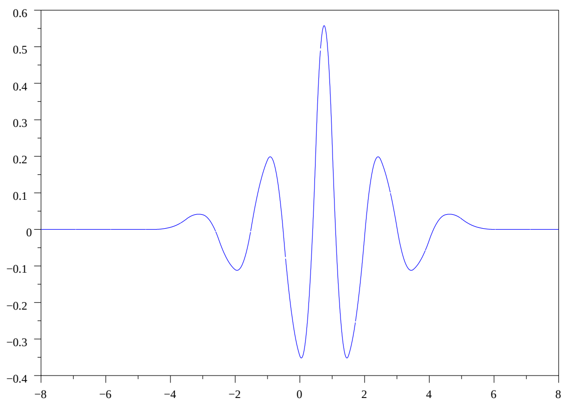
III - Numerical simulations, results

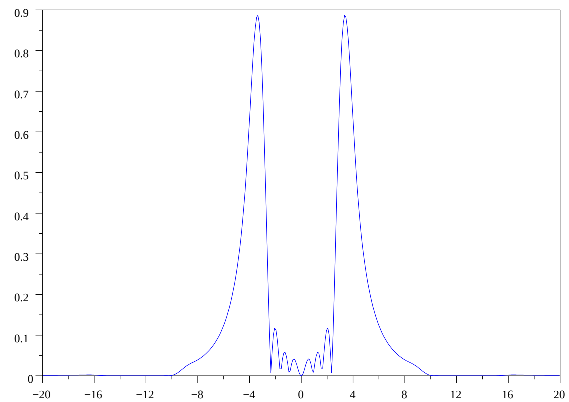
Conclusion - Perspectives

Wavelets

[Meyer90,Daubechies92]

Family of functions $\{\psi_\omega\}_{\omega\in\Omega}$ that forms a basis (sometime orthogonal) of a functionnal space ($L^2(\mathbb{R})$ for example).





Example of a wavelet with its frenquency localisation.

Multiresolution Analysis

Definition [Mallat87]: A Multiresolution Analysis of $L^2(\mathbb{R})$ is defined as a sequence of closed sub-spaces $(V_j)_{j\in\mathbb{Z}}$ verifying :

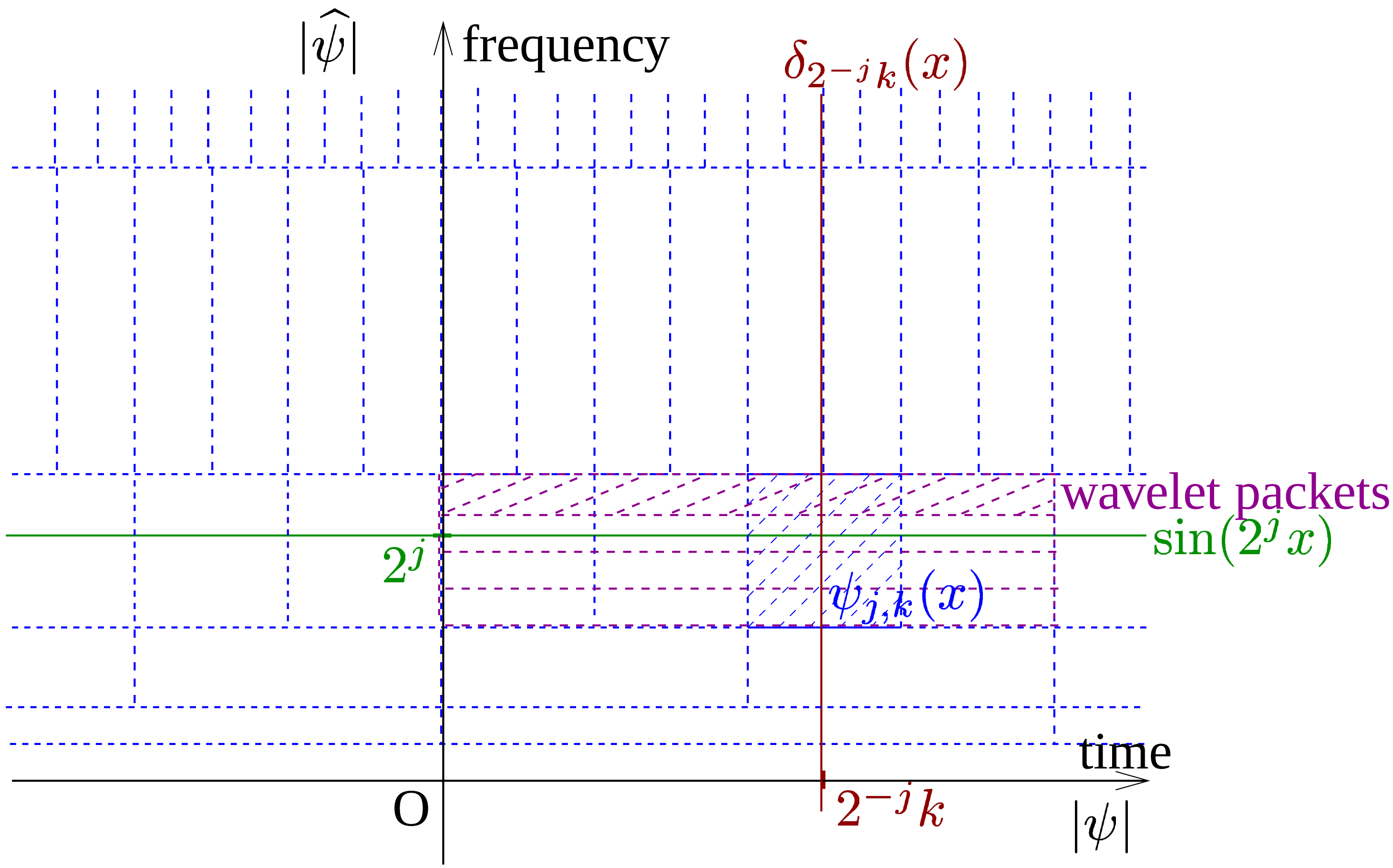
(1) $V_j \subset V_{j+1}, \quad \bigcap_{j\in\mathbb{Z}} V_j = \{0\} \quad \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$

(2) $f \in V_j \quad \Longleftrightarrow \quad f(2.) \in V_{j+1}$ (dilation)

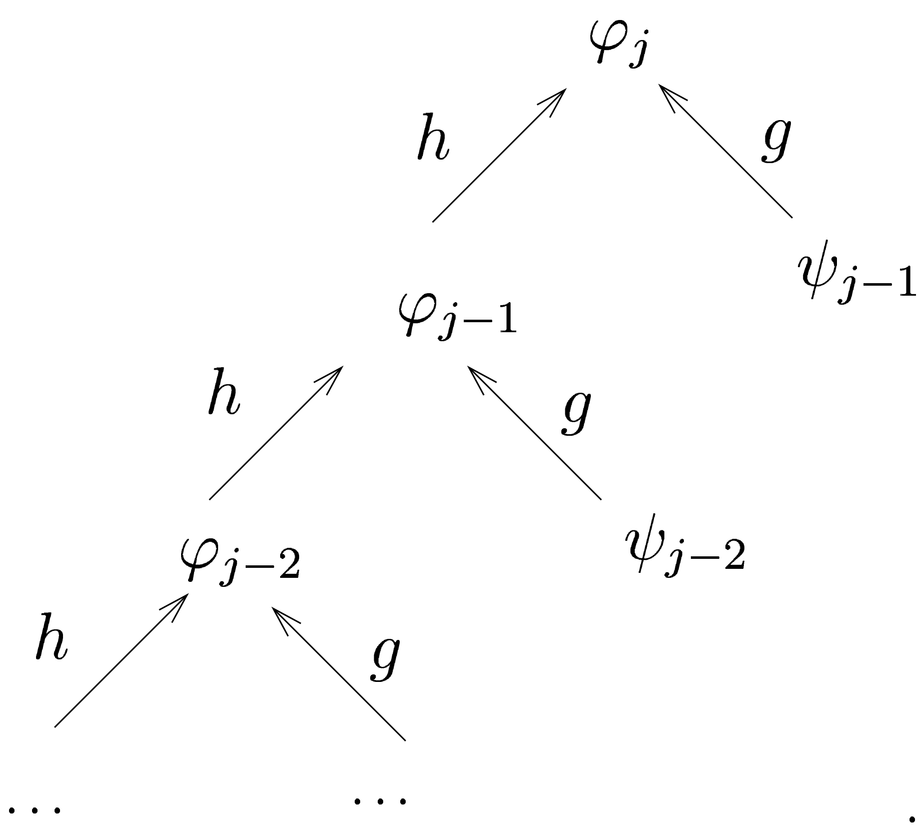
(3) $\exists \varphi \quad / \quad \{\varphi(. - k)\}_{k\in\mathbb{Z}}$ Riesz basis of V_0.

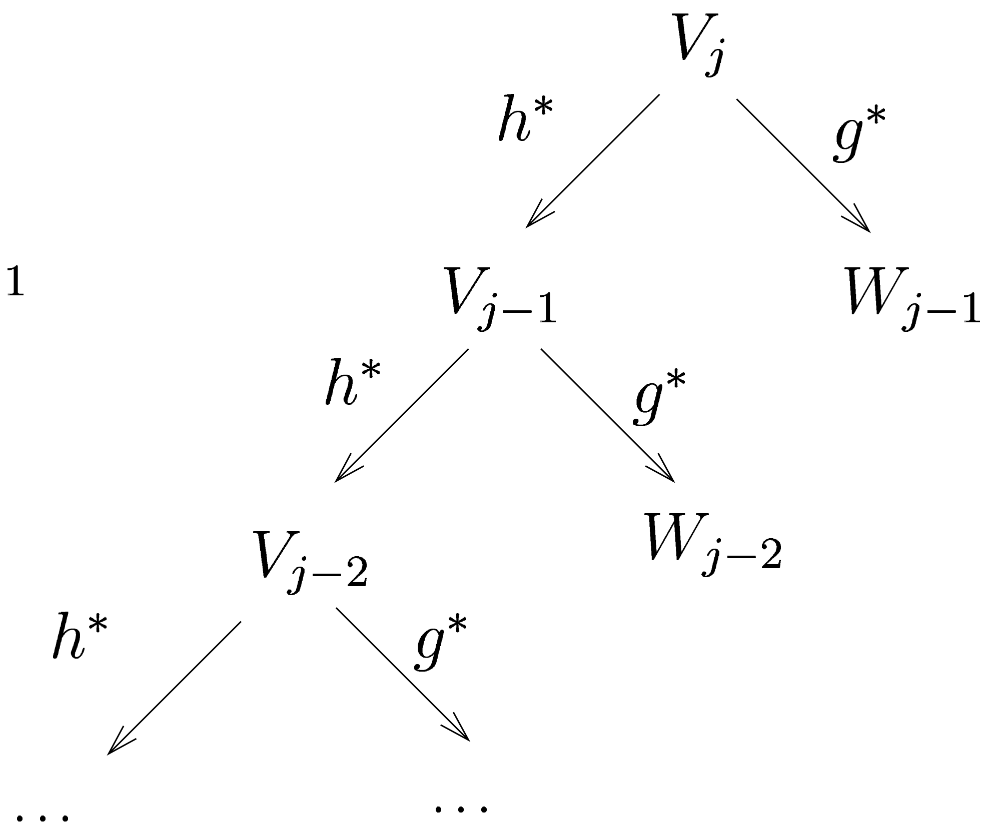
Wavelet space W_j defined by : $V_{j+1} = V_j \oplus^{\perp} W_j$.

Time-frequency partition with wavelets



Filtering Schema: decomposition – recomposition





Divergence-free wavelets

Proposition (Malgouyres): [Lemarié92] Let (φ_1, ψ_1) be an MRA. If $\varphi_1 \in C^{1+\epsilon}$ for a certain $\epsilon > 0$, then there is an MRA (φ_0, ψ_0) such that:

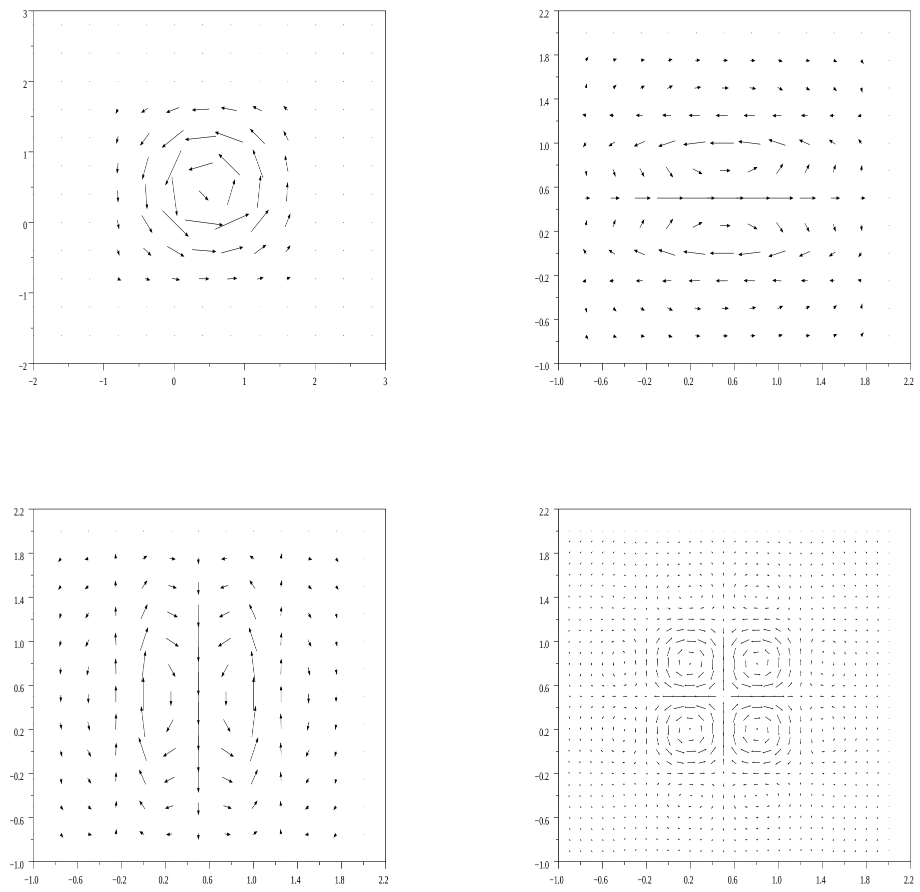
$$\varphi_1'(x) = \varphi_0(x) - \varphi_0(x-1)$$

$\text{wavelets}: \psi_0(x) = \frac{1}{4}\psi_1'(x)$

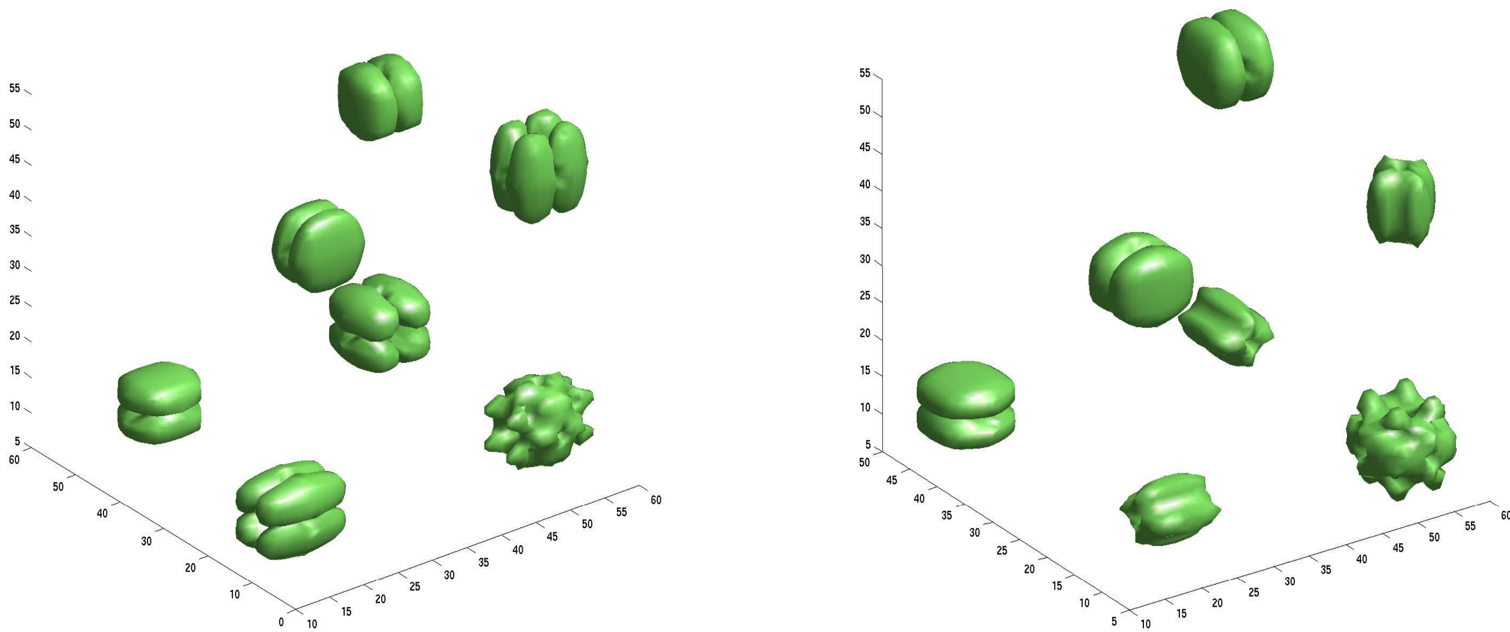
Theorem: There exist $(d-1)(2^d-1)$ vectorial functions $\overrightarrow{\Gamma_{\omega,i}} = (\Gamma_{\omega,i,1}; \ldots; \Gamma_{\omega,i,d})$, $\omega \in E^*$ and $i \neq i_0$ that, when translated and dilated, form an inconditionnelle basis of $\mathbf{H}_{div,0}$.

($E = \{0,1\}^d$, Γ constructed by tensor products of φ_0, φ_1, ψ_0 and ψ_1)

Example of divergence-free wavelets in 2D



Example of divergence-free wavelets in 3D



Vorticity isosurfaces of the 3D isotropic divergence-free wavelets

Divergence-free wavelet transform

$$\Psi_{df}^{(1,1)}(x_1,x_2) = \left| \begin{array}{l} \psi_1(x_1)\psi_0(x_2) \\ -\psi_0(x_1)\psi_1(x_2) \end{array} \right. \qquad \Psi_{cf}^{(1,1)}(x_1,x_2) = \left| \begin{array}{l} \psi_1(x_1)\psi_0(x_2) \\ \psi_0(x_1)\psi_1(x_2) \end{array} \right.$$

$$\left.\begin{array}{lcr} & \text{Standard wavelet transform} & \\ u_1 & \longrightarrow & d^{\varepsilon}_{1\ j,\mathbf{k}} \\ u_2 & \longrightarrow & d^{\varepsilon}_{2\ j,\mathbf{k}} \end{array}\right\} \begin{array}{c} \text{linear combinations} \\ \longrightarrow \end{array} \begin{array}{l} d^{\varepsilon}_{df\ j,\mathbf{k}} \\ d^{\varepsilon}_{cf\ j,\mathbf{k}} \end{array}$$

Anisotropic divergence-free wavelets

$$\Psi^{\text{div}}_{\mathbf{j},\mathbf{k}}(x_1,x_2) = \left| \begin{array}{l} 2^{j_2}\psi_1(2^{j_1}x_1-k_1)\psi_0(2^{j_2}x_2-k_2) \\ -2^{j_1}\psi_0(2^{j_1}x_1-k_1)\psi_1(2^{j_2}x_2-k_2) \end{array} \right.$$

with $\mathbf{j}=(j_1,j_2)\in\mathbb{Z}^2$ the scale and $\mathbf{k}=(k_1,k_2)\in\mathbb{Z}^2$ the position.

The operation on the coefficients:

$$\begin{bmatrix} d^{\text{div}}_{\mathbf{j},\mathbf{k}} \\ d^{\text{n}}_{\mathbf{j},\mathbf{k}} \end{bmatrix} = \frac{1}{2^{2j_1}+2^{2j_2}} \begin{bmatrix} 2^{j_2} & -2^{j_1} \\ 2^{j_1} & 2^{j_2} \end{bmatrix} \begin{bmatrix} d_{1\,\mathbf{j},\mathbf{k}} \\ d_{2\,\mathbf{j},\mathbf{k}} \end{bmatrix}$$

is an orthogonal basis change (orthogonal matrix).

Anisotropic divergence-free wavelets in n-D:

$$\Psi^{\mathrm{div}}_{i\,\mathbf{j},\mathbf{k}}(x_1,\ldots,x_n) = \left| \begin{array}{l} -2^{j_i+j_1}\prod_\ell \psi_{\delta_{1,\ell}}(2^{j_\ell}x_\ell - k_\ell) \\ \vdots \\ \left(\sum_{\ell\neq i} 2^{2j_\ell}\right)\prod_\ell \psi_{\delta_{i,\ell}}(2^{j_\ell}x_\ell - k_\ell) \\ \vdots \\ -2^{j_i+j_n}\prod_\ell \psi_{\delta_{n,\ell}}(2^{j_\ell}x_\ell - k_\ell) \end{array} \right.$$

$$\Psi^{\mathrm{n}}_{i\,\mathbf{j},\mathbf{k}}(x_1,\ldots,x_n) = \left| \begin{array}{l} 2^{j_1}\prod_\ell \psi_{\delta_{1,\ell}}(2^{j_\ell}x_\ell - k_\ell) \\ \vdots \\ 2^{j_n}\prod_\ell \psi_{\delta_{n,\ell}}(2^{j_\ell}x_\ell - k_\ell) \end{array} \right.$$

Let $\xi_\ell = 2^{j_\ell}$ and $|\xi|^2 = \sum_{\ell=1}^{n} 2^{2j_\ell}$,

$$\begin{bmatrix} d^{\mathrm{div}}_{1\,\mathbf{j},\mathbf{k}} \\ d^{\mathrm{div}}_{2\,\mathbf{j},\mathbf{k}} \\ \vdots \\ \vdots \\ d^{\mathrm{div}}_{n\,\mathbf{j},\mathbf{k}} \\ d^{\mathrm{n}}_{\mathbf{j},\mathbf{k}} \end{bmatrix} = \begin{bmatrix} 1-\frac{\xi_1^2}{|\xi|^2} & -\frac{\xi_2\xi_1}{|\xi|^2} & \cdots & \cdots & -\frac{\xi_n\xi_1}{|\xi|^2} & \frac{\xi_1}{|\xi|} \\ -\frac{\xi_1\xi_2}{|\xi|^2} & 1-\frac{\xi_2^2}{|\xi|^2} & \ddots & \ddots & -\frac{\xi_n\xi_2}{|\xi|^2} & \frac{\xi_2}{|\xi|} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ -\frac{\xi_1\xi_n}{|\xi|^2} & -\frac{\xi_2\xi_n}{|\xi|^2} & \cdots & -\frac{\xi_{n-1}\xi_n}{|\xi|^2} & 1-\frac{\xi_n^2}{|\xi|^2} & \frac{\xi_n}{|\xi|} \\ \frac{\xi_1}{|\xi|} & \frac{\xi_2}{|\xi|} & \cdots & \frac{\xi_{n-1}}{|\xi|} & \frac{\xi_n}{|\xi|} & 0 \end{bmatrix} \begin{bmatrix} d_{1\,\mathbf{j},\mathbf{k}} \\ d_{2\,\mathbf{j},\mathbf{k}} \\ \vdots \\ \vdots \\ d_{n\,\mathbf{j},\mathbf{k}} \\ 0 \end{bmatrix}$$

Matrix of size $(n+1)\times(n+1)$, orthogonal.

II - Helmholtz decomposition

Principal

Vector field $\mathbf{u} \in (L^2(\mathbb{R}^n))^n$, decomposition with

$$\mathbf{u} = \mathbf{u}_{\mathrm{div}} + \mathbf{u}_{\mathrm{curl}} \quad \text{where} \quad \mathbf{u}_{\mathrm{div}} = \mathbf{curl}\,\psi \quad \mathbf{u}_{\mathrm{curl}} = \nabla p$$

the functions $\mathbf{curl}\,\psi$ and ∇p are orthogonal in $(L^2(\mathbb{R}^n))^n$ and we have uniqueness.

$$(L^2(\mathbb{R}^n))^n = \mathbf{H}_{\mathrm{div}\,0}(\mathbb{R}^n) \oplus^{\perp} \mathbf{H}_{\mathrm{curl},0}(\mathbb{R}^n)$$

In N-S, importance of this decomposition to project the term $\mathbf{u}.\nabla\mathbf{u}$ onto $\mathbf{H}_{\mathrm{div}\,0}(\mathbb{R}^n)$.

Leray projector (in Fourier)

$$\mathbf{u} = \mathbf{u}_{\mathrm{div}} + \nabla p$$

$$\Delta p = \mathrm{div}\mathbf{u} = \partial_1 u_1 + \ldots + \partial_n u_n$$

In Fourier,

$$\hat{p} = -\frac{i}{|\xi|^2} \sum_{\ell=1}^{n} \xi_\ell \hat{u}_\ell$$

et

$$\hat{\mathbf{u}}_{\mathrm{div}} = \hat{\mathbf{u}} - \begin{bmatrix} \xi_1 \sum_{\ell=1}^n \xi_\ell \hat{u}_\ell \\ \xi_2 \sum_{\ell=1}^n \xi_\ell \hat{u}_\ell \\ \vdots \\ \xi_n \sum_{\ell=1}^n \xi_\ell \hat{u}_\ell \end{bmatrix}$$

Wavelet Helmholtz decomposition

We want to write:

$$\mathbf{v} = \mathbf{v}_{\mathrm{div}} + \mathbf{v}_{\mathrm{curl}}$$

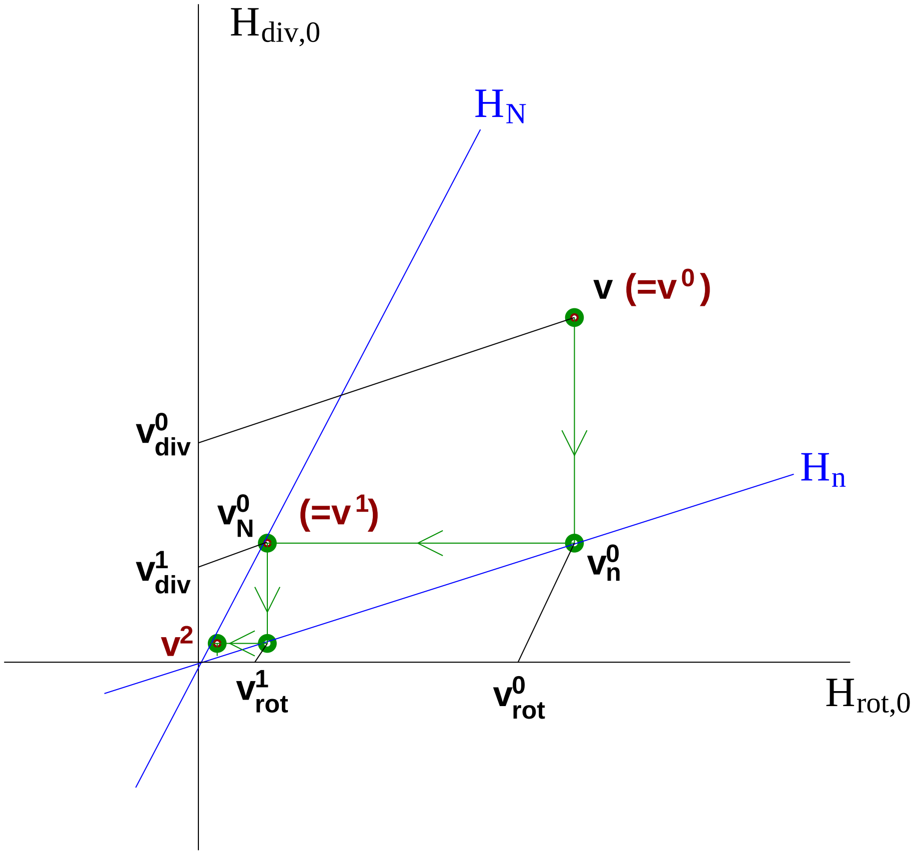
with

$$\mathbf{v}_{\mathrm{div}} = \sum_{\mathbf{j},\mathbf{k}} d_{\mathrm{div}\,\mathbf{j},\mathbf{k}} \Psi_{\mathrm{div}\,\mathbf{j},\mathbf{k}} \qquad \text{and} \qquad \mathbf{v}_{\mathrm{curl}} = \sum_{\mathbf{j},\mathbf{k}} d_{\mathrm{curl}\,\mathbf{j},\mathbf{k}} \Psi_{\mathrm{curl}\,\mathbf{j},\mathbf{k}}$$

Problem : the projectors on the divergence-free wavelet basis and on the gradient wavelet basis are *biorthogonal* projectors.

$\rightarrow$ Iterative method to find $\mathbf{v}_{\mathrm{div}}$ et $\mathbf{v}_{\mathrm{curl}}$.

Construction of the sequences $\mathbf{v}_{\mathrm{div}}^{p}$ and $\mathbf{v}_{\mathrm{curl}}^{p}$



Convergence processus for the sequences with $\mathrm{H_N} = vect\{\Psi^{\mathrm{N}}_{\mathbf{j},\mathbf{k}}\}$ *and* $\mathrm{H_n} = vect\{\Psi^{\mathrm{n}}_{\mathbf{j},\mathbf{k}}\}$.

Theorem: Convergence in dimension 2 for Shannon wavelets.

Proof (Kai Bittner): Looking at the proximity of H_{n} to $\mathrm{H}_{\mathrm{rot},0}$, we find a convergence criteria.

If there are $q_{\mathrm{n}}, q_{\mathrm{N}} \in \mathbb{R}$ such that:

$$\forall \mathbf{f}_{\mathrm{n}} \in \mathrm{H}_{\mathrm{n}}, \;\; \|\mathbb{P}\,\mathbf{f}_{\mathrm{n}}\|_{L^2} \leq q_{\mathrm{n}}\,\|\mathbb{Q}\,\mathbf{f}_{\mathrm{n}}\|_{L^2} \tag{1}$$

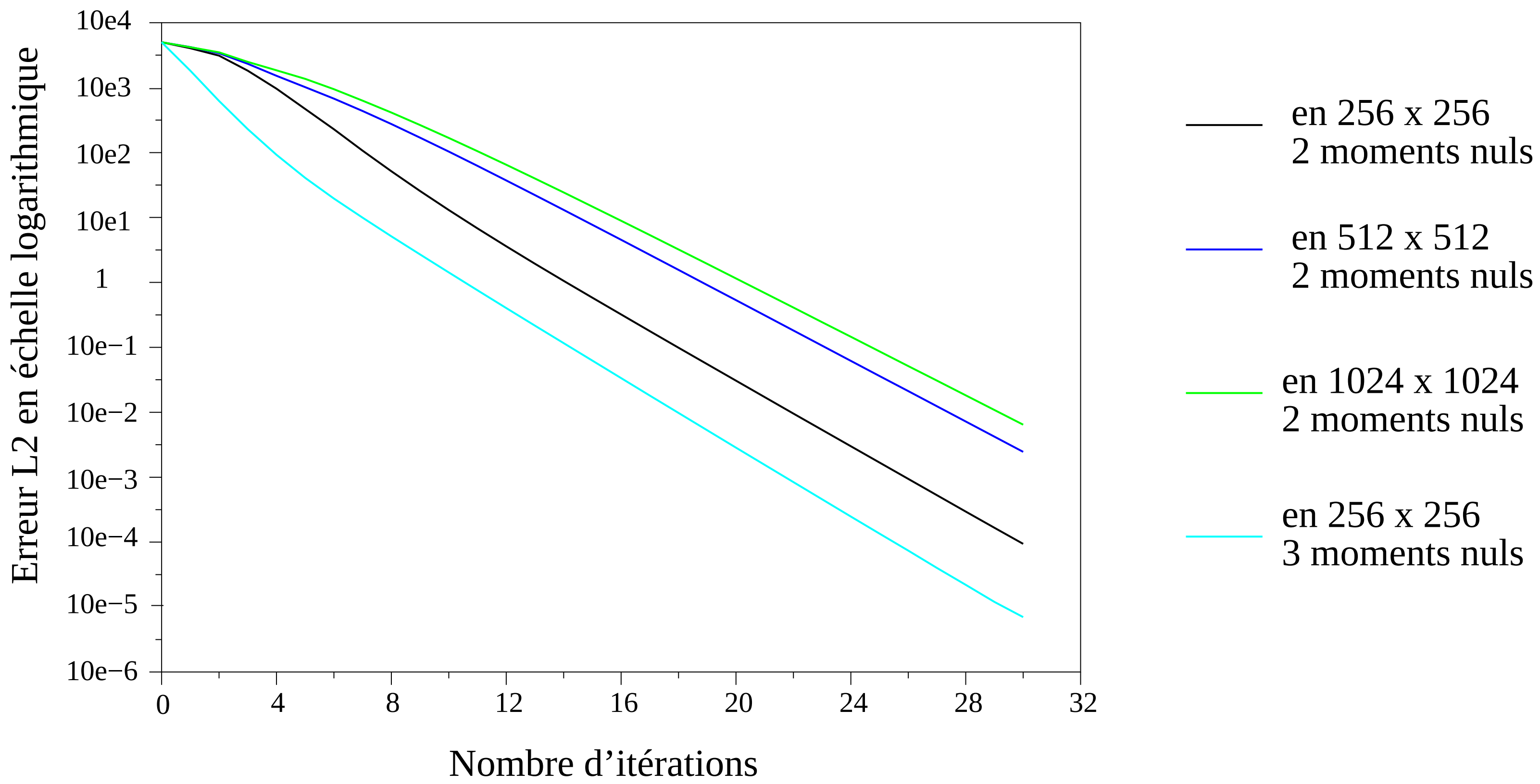
and

$$\forall \mathbf{f}_{\mathrm{N}} \in \mathrm{H}_{\mathrm{N}}, \;\; \|\mathbb{Q}\,\mathbf{f}_{\mathrm{N}}\|_{L^2} \leq q_{\mathrm{N}}\,\|\mathbb{P}\,\mathbf{f}_{\mathrm{N}}\|_{L^2} \tag{2}$$

then

$$\|\mathbf{v}^{p+1}\|_{L^2} \leq q_{\mathrm{n}}\,q_{\mathrm{N}}\|\mathbf{v}^{p}\|_{L^2} \tag{3}$$

Numericaly the convergence have been tested successfully on variate 2D and 3D fields.



Problem of the frequency localisation of the wavelets

Convergence rate linked ($\sim$ proportional) to :

$$\rho = \iint_{\xi \in \mathbb{R}^2} \frac{(\xi_1^2 - \xi_2^2)^2}{(\xi_1^2 + \xi_2^2)^2} \left| \widehat{\psi_1}(\xi_1)\, \widehat{\psi_1}(\xi_2) \right|^2 \, d\xi$$

▶ Problem on the frequency localisation of the wavelets.

- in Fourier

- ponderation function:

$$\omega(\xi) = \frac{(\xi_1^2 - \xi_2^2)^2}{(\xi_1^2 + \xi_2^2)^2}$$

- Size of the compact support:

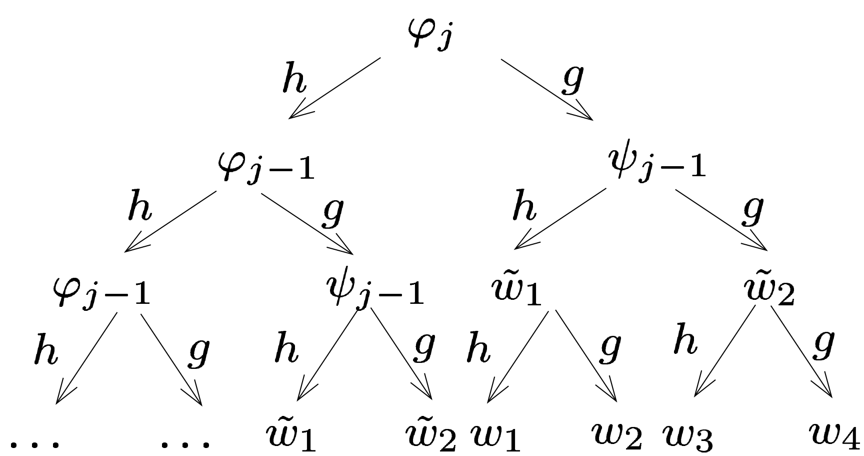
$$\left| \widehat{\psi_1}(\xi_1)\, \widehat{\psi_1}(\xi_2) \right|$$

Conclusion : we must get a better localisation in frequence for ψ_1.

The wavelet packets

Definition : $w_n(x)$, $n \in \mathbb{N}$, packets associated to the scale function φ :

$$\hat{w}_n(\xi) = \prod_{j=1}^{N} m_{\epsilon_j}\left(\frac{\xi}{2^j}\right)\hat{\varphi}\left(\frac{\xi}{2^N}\right), \quad n = \sum_{j=1}^{N} \epsilon_j 2^{j-1}, \quad \epsilon_j \in \{0, 1\}.$$



In general, fail to control the frequency localisation.

Fequency target

- With the Shannon wavelets

- with the Walsh packets

By iteration, $w_\omega \sim \cos(2\pi\omega \cdot +\theta)$ for

$$\omega = \sum_{j \geq 1} \epsilon_j 2^{-j}$$

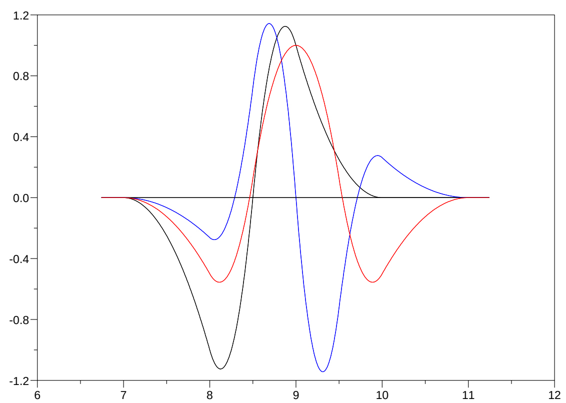
Packets modulation

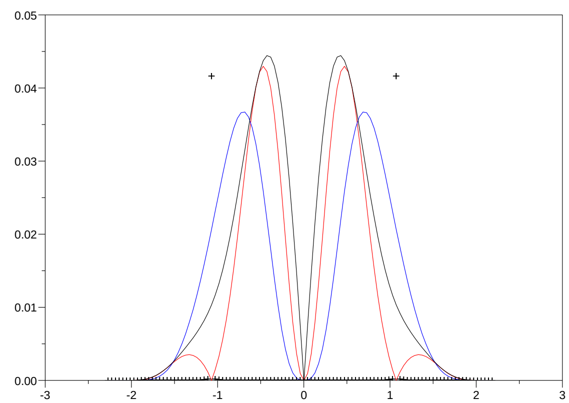
"A theoretical study show that we have to target"

Ideal Packet :

$$\psi_\omega(x) = \sum_{k\in\mathbb{Z}} \cos(2\pi\omega k + \theta)\varphi(2x - k)$$

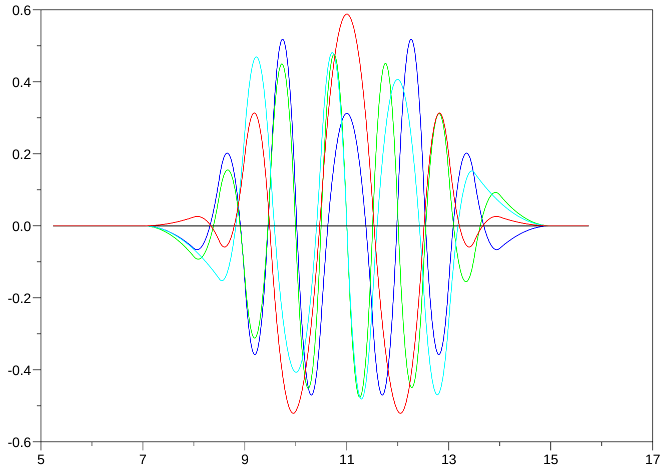
Examples :

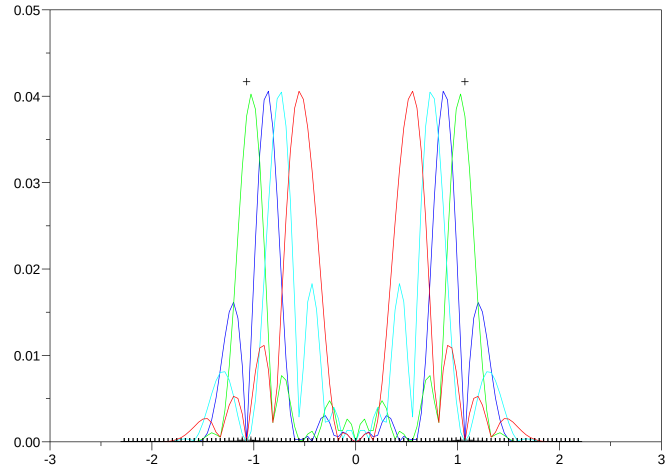




Quadratic spline wavelet packets with 2 wavelets

Packets with 4 quadratic spline wavelets





Numerical schema for Navier-Stokes

$\mathbb{P}$ Leray projector with wavelets (give the pressure directly) :
$\mathbb{P}\left[(\mathbf{u}\cdot\nabla)\mathbf{u}\right] = (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla p$

Δ operator is linear.

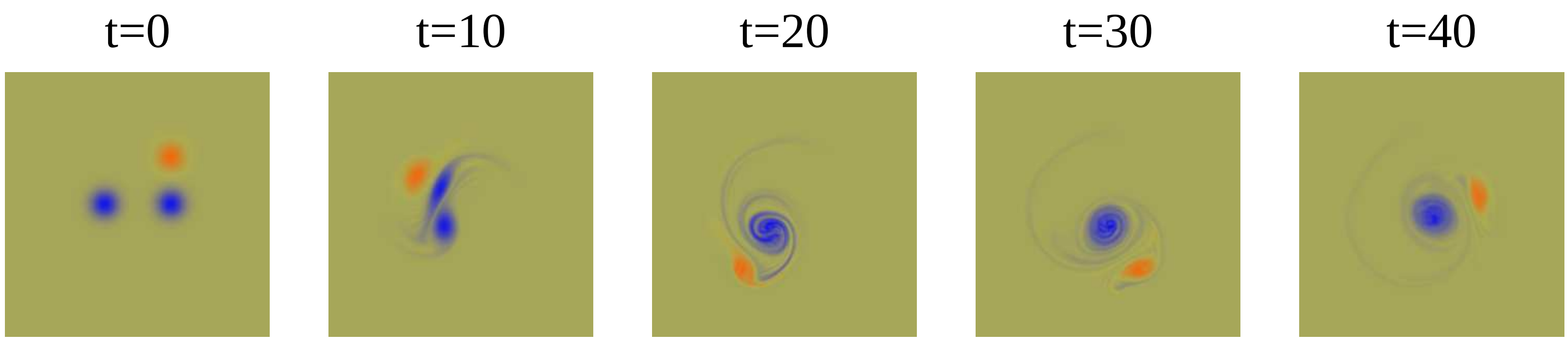
Euler explicite in time :

$$\mathbf{u}^{n+1} = \mathbf{u}^{n} - \delta t \mathbb{P}\left[\mathbf{u}.\nabla\mathbf{u}\right]^{n} + \delta t \nu \boldsymbol{\Delta}\mathbf{u}^{n}$$

on the wavelet coefficients :

$$d_{i\,\mathbf{j},\mathbf{k}}^{\mathrm{div},n+1} = d_{i\,\mathbf{j},\mathbf{k}}^{\mathrm{div},n} - \delta t d_{i\,\mathbf{j},\mathbf{k}}^{\mathrm{div}}(\mathbb{P}\left[(\mathbf{u}\cdot\nabla)\mathbf{u}\right]) + \delta t \nu d_{i\,\mathbf{j},\mathbf{k}}^{\mathrm{div}}(\boldsymbol{\Delta}\mathbf{u}^{n})$$

Test with the simulation “fusion of 3 vortices”



- wavelet code splines of degree 1 and 2 the simplest ($\sim$ 30 iterations for Helmholtz)

- *Runge-Kutta* schema of order 2 for the time evolution

- 256^2 grid

Results are visually identical to a spectral code in 256^2.

Conclusion

Assets

- Calculation in $O(n)$
- Scale separation
- Non linear approximation

Perspectives

- Get adaptativity
- Limit conditions
- Complex geometries